# Convex Optimization for Signal Processing and Communications: 

## From Fundamentals to Applications

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## Outline

(1) Part I: Fundamentals of Convex Optimization
(2) Part II: Application in Hyperspectral Image Analysis: (Big Data Analysis and Machine Learning)
(3) Part III: Application in Wireless Communications (5G Systems)

- Subsection I: Outage Constrained Robust Transmit Optimization for Multiuser MISO Downlinks
- Subsection II: Outage Constrained Robust Hybrid Coordinated Beamforming for Massive MIMO Enabled Heterogeneous Cellular Networks


## Optimization problem

- Optimization problem:

$$
\begin{align*}
\operatorname{minimize} & f(\mathbf{x}) \\
\text { subject to } & \mathbf{x} \in \mathcal{C} \tag{1}
\end{align*}
$$

where $f(\mathbf{x})$ is the objective function to be minimized and $\mathcal{C}$ is the feasible set from which we try to find an optimal solution. Let

$$
\begin{equation*}
\mathbf{x}^{\star}=\arg \min _{\mathbf{x} \in \mathcal{C}} f(\mathbf{x}) \quad \text { (optimal solution or global minimizer) } \tag{2}
\end{equation*}
$$

- Challenges in applications:
- Local optima; large problem size; decision variable $\mathbf{x}$ involving real and/or complex vectors, matrices; feasible set $\mathcal{C}$ involving generalized inequalities, etc.
- Computational complexity: NP-hard; polynomial-time solvable.
- Performance analysis: Performance insights, properties, perspectives, proofs (e.g., identifiability and convergence), limits and bounds.


## Convex sets and convex functions-1

- Affine (convex) combination: Provided that $C$ is a nonempty set,

$$
\begin{equation*}
\mathbf{x}=\theta_{1} \mathbf{x}_{1}+\cdots+\theta_{K} \mathbf{x}_{K}, \mathbf{x}_{i} \in C \forall i \tag{3}
\end{equation*}
$$

is called an affine (a convex) combination of $\mathbf{x}_{1}, \ldots, \mathbf{x}_{K}$ ( $K$ vectors or points of a set) if $\sum_{i=1}^{K} \theta_{i}=1, \theta_{i} \in \mathbb{R}\left(\theta_{i} \in \mathbb{R}_{+}\right), K \in \mathbb{Z}_{++}$.

- Affine (convex) set:
- $C$ is an affine (a convex) set if $C$ is closed under the operation of affine (convex) combination;
- an affine set is constructed by lines;
- a convex set is constructed by line segments.
- Conic set:
- If $\theta \mathbf{x} \in C$ for any $\theta \in \mathbb{R}_{+}$and any $\mathbf{x} \in C$, then the set $C$ is a cone and it is constructed by rays starting from the origin;
- the linear combination (3) is called a conic combination if $\theta_{i} \geq 0 \forall i$;


## Convex Set Examples



Ellipsoid and Euclidean ball centered at $\mathbf{x}_{c}$


Second-order cone

$$
C=\left\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} \mid\|\mathbf{x}\|_{2} \leq t\right\}
$$

- Left plot: An ellipsoid with semiaxes $\sqrt{\lambda_{1}}, \sqrt{\lambda_{2}}$, and an Euclidean ball with radius $r>\max \left\{\sqrt{\lambda_{1}}, \sqrt{\lambda_{2}}\right\}$ in $\mathbb{R}^{2}$; right plot: Second-order cone in $\mathbb{R}^{3}$.


## Convex sets and convex functions-2

- Let $\mathcal{A}=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}\right\} \subset \mathbb{R}^{M}$. Affine hull of $\mathcal{A}$ (the smallest affine set containing $\mathcal{A}$ ) is defined as

$$
\text { aff } \begin{aligned}
\mathcal{A} & \triangleq\left\{\mathbf{x}=\sum_{i=1}^{N} \theta_{i} \mathbf{a}_{i} \mid \sum_{i=1}^{N} \theta_{i}=1, \theta_{i} \in \mathbb{R} \forall i\right\} \\
& =\left\{\mathbf{x}=\mathbf{C} \boldsymbol{\alpha}+\mathbf{d} \mid \boldsymbol{\alpha} \in \mathbb{R}^{p}\right\} \quad \text { (affine set representation) }
\end{aligned}
$$

where $\mathbf{C} \in \mathbb{R}^{M \times p}$ is of full column rank, $\mathbf{d} \in \operatorname{aff} \mathcal{A}$, and

$$
\operatorname{affdim} \mathcal{A}=p \leq \min \{N-1, M\}
$$

- $\mathcal{A}$ is affinely independent $(\mathcal{A} . \mathcal{I}$.) with $\operatorname{affdim} \mathcal{A}=N-1$ if the set $\left\{\mathbf{a}-\mathbf{a}_{i} \mid \mathbf{a} \in \mathcal{A}, \mathbf{a} \neq \mathbf{a}_{i}\right\}$ is linearly independent for any $i$; moreover,

$$
\text { aff } \left.\mathcal{A}=\left\{\mathbf{x} \mid \mathbf{b}^{T} \mathbf{x}=h\right\} \triangleq \mathcal{H}(\mathbf{b}, h) \quad \text { (when } M=N\right)
$$

is a hyperplane, where $(\mathbf{b}, h)$ can be determined from $\mathcal{A}$ (closed-form expressions available).

## Convex sets and convex functions-3



Figure 1: An illustration in $\mathbb{R}^{3}$, where $\operatorname{conv}\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}\right\}$ is a simplex defined by the shaded triangle, and $\operatorname{conv}\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \mathbf{a}_{4}\right\}$ is a simplex (and also a simplest simplex) defined by the tetrahedron with the four extreme points $\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \mathbf{a}_{4}\right\}$.

## Convex sets and convex functions-4

- Let $\mathcal{A}=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}\right\} \subset \mathbb{R}^{M}$. Convex hull of $\mathcal{A}$ (the smallest convex set containing $\mathcal{A}$ ) is defined as

$$
\operatorname{conv} \mathcal{A}=\left\{\mathbf{x}=\sum_{i=1}^{N} \theta_{i} \mathbf{a}_{i} \mid \sum_{i=1}^{N} \theta_{i}=1, \theta_{i} \in \mathbb{R}_{+} \forall i\right\} \subset \operatorname{aff} \mathcal{A}
$$

- $\operatorname{conv} \mathcal{A}$ is called a simplex (a polytope with $N$ vertices) if $\mathcal{A}$ is $\mathcal{A}$.I.
- When $\mathcal{A}$ is $\mathcal{A}$.I. and $M=N-1$, conv $\mathcal{A}$ is called a simplest simplex of $N$ vertices (i.e., $\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}$ );

$$
\text { aff }\left(\mathcal{A} \backslash\left\{\mathbf{a}_{i}\right\}\right)=\mathcal{H}\left(\mathbf{b}_{i}, h_{i}\right), i \in \mathcal{I}_{N}=\{1, \ldots, N\}
$$

is a hyperplane, where the $N$ boundary hyperplane parameters $\left\{\left(\mathbf{b}_{i}, h_{i}\right), i=1, \ldots, N\right\}$ can be uniquely determined from $\mathcal{A}$ (closed-form expressions available) and vice versa.

## Convex Set Examples



- Left plot: conic $C$ (called the conic hull of $C$ ) is a convex cone formed by $C=\left\{\mathbf{x}_{1}, \mathbf{x}_{2}\right\}$ via conic combinations, i.e., the smallest conic set that contains $C$; right plot: conic $C$ formed by another set $C$ (star).


## Convex sets and convex functions-5




- Left plot: $\frac{\mathbf{y}_{1}+\mathbf{y}_{2}}{2} \notin B$, implying that $B$ is not a convex set; right plot: $f(x)$ is a convex function (by (4)).


## Convex sets and convex functions-6

- Convex function: $f$ is convex if $\operatorname{dom} f$ (the domain of $f$ ) is a convex set, and for all $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$,

$$
\begin{equation*}
f(\theta \mathbf{x}+(1-\theta) \mathbf{y}) \leq \theta f(\mathbf{x})+(1-\theta) f(\mathbf{y}), \forall 0 \leq \theta \leq 1 \tag{4}
\end{equation*}
$$

- $f$ is concave if $-f$ is convex.


## Some Examples of Convex Functions

- An affine function $f(\mathbf{x})=\mathbf{a}^{T} \mathbf{x}+b$ is both convex and concave on $\mathbb{R}^{n}$.
- $f(\mathbf{x})=\mathbf{x}^{T} \mathbf{P} \mathbf{x}+2 \mathbf{q}^{T} \mathbf{x}+r$, where $\mathbf{P} \in \mathbb{S}^{n}, \mathbf{q} \in \mathbb{R}^{n}$ and $r \in \mathbb{R}$ is convex if and only if $\mathbf{P} \in \mathbb{S}_{+}^{n}$.
- Every norm on $\mathbb{R}^{n}$ (e.g., $\|\cdot\|_{p}$ for $p \geq 1$ ) is convex.
- Linear function $f(\mathbf{X})=\operatorname{Tr}(\mathbf{A X})$ (where $\operatorname{Tr}(\mathbf{V})$ denotes the trace of a square matrix $\mathbf{V}$ ) is both convex and concave on $\mathbb{R}^{n \times n}$.
- $f(\mathbf{X})=-\log \operatorname{det}(\mathbf{X})$ is convex on $\mathbb{S}_{++}^{n}$.


## Ways Proving Convexity of a Function

## First-order Condition

Suppose that $f$ is differentiable. $f$ is a convex function if and only if $\operatorname{dom} f$ is a convex set and

$$
\begin{equation*}
f(\mathbf{y}) \geq f(\mathbf{x})+\nabla f(\mathbf{x})^{T}(\mathbf{y}-\mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in \operatorname{dom} f \tag{5}
\end{equation*}
$$

## Second-order Condition

Suppose that $f$ is twice differentiable. $f$ is a convex function if and only if $\operatorname{dom} f$ is a convex set and

$$
\begin{equation*}
\nabla^{2} f(\mathbf{x}) \succeq \mathbf{0} \text { (positive-semidefinite), } \forall \mathbf{x} \in \operatorname{dom} f \tag{6}
\end{equation*}
$$

## Epigraph

The epigraph of a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is defined as

$$
\begin{equation*}
\text { epi } f=\{(\mathbf{x}, t) \mid \mathbf{x} \in \operatorname{dom} f, f(\mathbf{x}) \leq t\} \subseteq \mathbb{R}^{n+1} \tag{7}
\end{equation*}
$$

A function $f$ is convex if and only if epi $f$ is a convex set.

## First-order Condition and Epigraph



- Left plot: first-order condition for a convex function $f$ for the one-dimensional case: $f(b) \geq f(a)+f^{\prime}(a)(b-a)$, for all $a, b \in \operatorname{dom} f$; right plot: the epigraph of a convex function $f: \mathbb{R} \rightarrow \mathbb{R}$.


## Convex Functions (Cont'd)

## Convexity Preserving Operations

- Intersection, $\bigcap_{i} C_{i}$ of convex sets $C_{i}$ is a convex set;

Nonnegative weighted sum, $\sum_{i} \theta_{i} f_{i}$ (where $\theta_{i} \geq 0$ ) of convex functions $f_{i}$ is a convex function.

- Image, $h(C)=\{h(\mathbf{x}) \mid \mathbf{x} \in C)\}$, of a convex set $C$ via affine mapping $h(\mathbf{x}) \triangleq \mathbf{A x}+\mathbf{b}$, is a convex set;
Composition $f(h(\mathbf{x}))$ of a convex function $f$ with affine mapping $h$ is a convex function;
- Image, $p(C)$, of a convex set $C$ via perspective mapping $p(\mathbf{x}, t) \triangleq \mathbf{x} / t$ is a convex set;
Perspective, $g(\mathbf{x}, t)=t f(\mathbf{x} / t)($ where $t>0)$ of a convex function $f$ is a convex function.


## Perspective Mapping \& Perspective of a Function




- Left plot: pinhole camera interpretation of the perspective mapping $p(x, t)=x / t, t>0$;
right plot: epigraph of the perspective $g(x, t)=t f(x / t), t>0$ of $f(x)=x^{2}$, where each ray is associated with $g(a t, t)=a^{2} t$ for a different value of $a$.


## Convex optimization problem

- Convex problem:

$$
\begin{equation*}
(\mathrm{CVXP}) \quad p^{\star}=\min _{\mathbf{x} \in \mathcal{C}} f(\mathbf{x}) \tag{8}
\end{equation*}
$$

is a convex problem if the objective function $f(\cdot)$ is a convex function and $\mathcal{C}$ is a convex set (called the feasible set) in standard form as follows:

$$
\mathcal{C}=\left\{\mathbf{x} \in \mathcal{D} \mid f_{i}(\mathbf{x}) \leq 0, h_{j}(\mathbf{x})=0, i=1, \ldots, m, j=1, \ldots, p\right\}
$$

where $f_{i}(\mathbf{x})$ is convex for all $i$ and $h_{j}(\mathbf{x})$ is affine for all $j$ and

$$
\mathcal{D}=\operatorname{dom} f \cap\left\{\bigcap_{i=1}^{m} \operatorname{dom} f_{i}\right\} \bigcap\left\{\bigcap_{i=1}^{p} \operatorname{dom} h_{i}\right\}
$$

is called the problem domain.

## - Advantages:

- Global optimality: $\mathbf{x}^{\star}$ can be obtained by closed-form solution, analytically (algorithm), or numerically by convex solvers (e.g., CVX and SeDuMi).
- Computational complexity: Polynomial-time solvable.
- Performance analysis: KKT conditions are the backbone for analysis.


## Global optimality and solution

- An optimality criterion: Any suboptimal solution to CVXP (8) is globally optimal. Assume that $f$ is differentiable. Then a point $\mathbf{x}^{\star} \in \mathcal{C}$ is optimal if and only if

$$
\begin{equation*}
\nabla f\left(\mathbf{x}^{\star}\right)^{T}\left(\mathbf{x}-\mathbf{x}^{\star}\right) \geq 0, \forall \mathbf{x} \in \mathcal{C} \tag{9}
\end{equation*}
$$



Case 1: $\nabla f\left(\mathbf{x}^{\star}\right)=\mathbf{0}_{n}$
Case 2: $\nabla f\left(\mathbf{x}^{\star}\right)^{T}\left(\mathbf{x}-\mathbf{x}^{\star}\right) \geq 0, \forall \mathbf{x} \in \mathcal{C}$
(where int $\mathcal{C} \neq \emptyset$ is assumed)

## Global optimality and solution

- Besides the optimality criterion (9), a complementary approach for solving CVXP (8) is founded on the "duality theory".
- Dual problem:

$$
\begin{align*}
& \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) \triangleq f(\mathbf{x})+\sum_{i=1}^{m} \lambda_{i} f_{i}(\mathbf{x})+\sum_{i=1}^{p} \nu_{i} h_{i}(\mathbf{x}) \quad \text { (Lagrangian) } \\
& g(\boldsymbol{\lambda}, \boldsymbol{\nu})=\inf _{\mathbf{x} \in \mathcal{D}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu})>-\infty \quad \text { (dual function) } \\
& d^{\star}=\max \left\{g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \mid \boldsymbol{\lambda} \succeq \mathbf{0}, \boldsymbol{\nu} \in \mathbb{R}^{p}\right\} \quad \text { (dual problem) } \\
& \left.\quad \leq p^{\star}=\min \{f(\mathbf{x}) \mid \mathbf{x} \in \mathcal{C})\right\} \quad \text { (primal problem CVXP (8)) } \tag{10}
\end{align*}
$$

where $\boldsymbol{\lambda}=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ and $\boldsymbol{\nu}=\left(\nu_{1}, \ldots, \nu_{p}\right)$ are dual variables; " $\succeq$ " stands for an abbreviated generalized inequality defined by the proper cone $K=\mathbb{R}_{+}^{m}$, i.e., the first orthant, (a closed convex solid and pointed cone), i.e., $\boldsymbol{\lambda} \succeq_{K} \mathbf{0} \Leftrightarrow \boldsymbol{\lambda} \in K$.

CVXP (8) and its dual can be solved simultaneously by solving the so-called $K K T$ conditions.

## Global optimality and solution

## - KKT conditions:

Suppose that $f, f_{1}, \ldots, f_{m}, h_{1}, \ldots, h_{p}$ are differentiable and $\mathbf{x}^{\star}$ is primal optimal and $\left(\boldsymbol{\lambda}^{\star}, \boldsymbol{\nu}^{\star}\right)$ is dual optimal to CVXP (8). Under strong duality, i.e.,

$$
p^{\star}=d^{\star}=\mathcal{L}\left(\mathbf{x}^{\star}, \boldsymbol{\lambda}^{\star}, \boldsymbol{\nu}^{\star}\right)
$$

(which holds true under Slater's condition: a strictly feasible point exists, i.e., relint $\mathcal{C} \neq \emptyset)$, the KKT conditions for solving $\mathbf{x}^{\star}$ and $\left(\boldsymbol{\lambda}^{\star}, \boldsymbol{\nu}^{\star}\right)$ are as follows:

$$
\begin{align*}
\nabla_{\mathbf{x}} \mathcal{L}\left(\mathbf{x}^{\star}, \boldsymbol{\lambda}^{\star}, \boldsymbol{\nu}^{\star}\right) & =\mathbf{0}  \tag{11a}\\
f_{i}\left(\mathbf{x}^{\star}\right) & \leq 0, i=1, \ldots, m  \tag{11b}\\
h_{i}\left(\mathbf{x}^{\star}\right) & =0, i=1, \ldots, p  \tag{11c}\\
\lambda_{i}^{\star} & \geq 0, i=1, \ldots, m  \tag{11d}\\
\lambda_{i}^{\star} f_{i}\left(\mathbf{x}^{\star}\right) & =0, i=1, \ldots, m . \quad \text { (complementary slackness) } \tag{11e}
\end{align*}
$$

The above KKT conditions (11) and the optimality criterion (9) are equivalent under Slater's condition.

## Strong Duality



- Lagrangian $\mathcal{L}(x, \lambda)$, dual function $g(\lambda)$, and primal-dual optimal solution $\left(x^{\star}, \lambda^{\star}\right)=(1,1)$ of the convex problem $\min \left\{f_{0}(x)=x^{2} \mid(x-2)^{2} \leq 1\right\}$ with strong duality. Note that $f_{0}\left(x^{\star}\right)=g\left(\lambda^{\star}\right)=\mathcal{L}\left(x^{\star}, \lambda^{\star}\right)=1$.


## Standard Convex Optimization Problems

## Linear Programming (LP) - Inequality Form

$$
\begin{align*}
& \min \mathbf{c}^{T} \mathbf{x}  \tag{12}\\
& \text { s.t. } \mathbf{G x} \preceq \mathbf{h}, \quad(\preceq \text { stands for componentwise inequality }) \\
& \mathbf{A x}=\mathbf{b},
\end{align*}
$$

where $\mathbf{c} \in \mathbb{R}^{n}, \mathbf{A} \in \mathbb{R}^{p \times n}, \mathbf{b} \in \mathbb{R}^{p}, \mathbf{G} \in \mathbb{R}^{m \times n}, \mathbf{h} \in \mathbb{R}^{m}$, and $\mathbf{x} \in \mathbb{R}^{n}$ is the unknown vector variable.

## LP- Standard Form

$$
\begin{align*}
& \min \mathbf{c}^{T} \mathbf{x}  \tag{13}\\
& \text { s.t. } \mathbf{x} \succeq \mathbf{0} \\
& \mathbf{A} \mathbf{x}=\mathbf{b}
\end{align*}
$$

where $\mathbf{c} \in \mathbb{R}^{n}, \mathbf{A} \in \mathbb{R}^{p \times n}, \mathbf{b} \in \mathbb{R}^{p}$, and $\mathbf{x} \in \mathbb{R}^{n}$ is the unknown vector variable.

## Standard Convex Optimization Problems (Cont'd)

Quadratic Programming (QP): Convex if and only if $\mathbf{P} \succeq \mathbf{0}$ (i.e., $\mathbf{P}$ is positive semidefinite)

$$
\begin{align*}
\min & \frac{1}{2} \mathbf{x}^{T} \mathbf{P} \mathbf{x}+\mathbf{q}^{T} \mathbf{x}+r  \tag{14}\\
\text { s.t. } & \mathbf{A} \mathbf{x}=\mathbf{b}, \mathbf{G} \mathbf{x} \preceq \mathbf{h}
\end{align*}
$$

where $\mathbf{P} \in \mathbb{S}^{n}, \mathbf{G} \in \mathbb{R}^{m \times n}$, and $\mathbf{A} \in \mathbb{R}^{p \times n}$.

Quadratically constrained QP (QCQP): Convex if and only if $\mathbf{P}_{i} \succeq \mathbf{0}, \forall i$

$$
\begin{align*}
\min & \frac{1}{2} \mathbf{x}^{T} \mathbf{P}_{0} \mathbf{x}+\mathbf{q}_{0}^{T} \mathbf{x}+r_{0}  \tag{15}\\
\text { s.t. } & \frac{1}{2} \mathbf{x}^{T} \mathbf{P}_{i} \mathbf{x}+\mathbf{q}_{i}^{T} \mathbf{x}+r_{i} \leq 0, i=1, \ldots, m \\
& \mathbf{A} \mathbf{x}=\mathbf{b}
\end{align*}
$$

where $\mathbf{P}_{i} \in \mathbb{S}^{n}, i=0,1, \ldots, m$, and $\mathbf{A} \in \mathbb{R}^{p \times n}$.

## Standard Convex Optimization Problems (Cont'd)

## Second-order cone programming (SOCP)

$$
\begin{align*}
\min & \mathbf{c}^{T} \mathbf{x}  \tag{16}\\
\text { s.t. } & \left\|\mathbf{A}_{i} \mathbf{x}+\mathbf{b}_{i}\right\|_{2} \leq \mathbf{f}_{i}^{T} \mathbf{x}+d_{i}, \quad i=1, \ldots, m \\
& \mathbf{F} \mathbf{x}=\mathbf{g}
\end{align*}
$$

where $\mathbf{A}_{i} \in \mathbb{R}^{n_{i} \times n}, \mathbf{b}_{i} \in \mathbb{R}^{n_{i}}, \mathbf{f}_{i} \in \mathbb{R}^{n}, d_{i} \in \mathbb{R}, \mathbf{F} \in \mathbb{R}^{p \times n}, \mathbf{g} \in \mathbb{R}^{p}, \mathbf{c} \in \mathbb{R}^{n}$, and $\mathrm{x} \in \mathbb{R}^{n}$ is the vector variable.

## Semidefinite programming (SDP) - Standard Form

$$
\begin{align*}
\min & \operatorname{Tr}(\mathbf{C X})  \tag{17}\\
\text { s.t. } & \mathbf{X} \succeq \mathbf{0} \\
& \operatorname{Tr}\left(\mathbf{A}_{i} \mathbf{X}\right)=b_{i}, i=1, \ldots, p,
\end{align*}
$$

with variable $\mathbf{X} \in \mathbb{S}^{n}$, where $\mathbf{A}_{i} \in \mathbb{S}^{n}, \mathbf{C} \in \mathbb{S}^{n}$, and $b_{i} \in \mathbb{R}$.

## Alternating direction method of multiplers (ADMM)

- Consider the following convex optimization problem:

$$
\begin{align*}
\min _{\mathbf{x} \in \mathbb{R}^{n}, \mathbf{z} \in \mathbb{R}^{m}} & f_{1}(\mathbf{x})+f_{2}(\mathbf{z}) \\
\text { s.t. } & \mathbf{x} \in \mathcal{S}_{1}, \mathbf{z} \in \mathcal{S}_{2}  \tag{18}\\
& \mathbf{z}=\mathbf{A} \mathbf{x}
\end{align*}
$$

where $f_{1}: \mathbb{R}^{n} \mapsto \mathbb{R}$ and $f_{2}: \mathbb{R}^{m} \mapsto \mathbb{R}$ are convex functions, $\mathbf{A}$ is an $m \times n$ matrix, and $\mathcal{S}_{1} \subset \mathbb{R}^{n}$ and $\mathcal{S}_{2} \subset \mathbb{R}^{m}$ are nonempty convex sets.

- The considered dual problem of $(18)$ is given by

$$
\begin{equation*}
\max _{\boldsymbol{\nu} \in \mathbb{R}^{m}} \min _{\mathbf{x} \in \mathcal{S}_{1}, \mathbf{z} \in \mathcal{S}_{2}}\left\{f_{1}(\mathbf{x})+f_{2}(\mathbf{z})+\frac{c}{2}\|\mathbf{A} \mathbf{x}-\mathbf{z}\|_{2}^{2}+\boldsymbol{\nu}^{T}(\mathbf{A} \mathbf{x}-\mathbf{z})\right\} \tag{19}
\end{equation*}
$$

where $c$ is a penalty parameter, and $\boldsymbol{\nu}$ is the dual variable associated with the equality constraint in (18).

## ADMM (Cont'd)

- Inner minimization (convex problems):

$$
\begin{align*}
& \mathbf{z}(q+1)=\arg \min _{\mathbf{z} \in \mathcal{S}_{2}}\left\{f_{2}(\mathbf{z})-\boldsymbol{\nu}(q)^{T} \mathbf{z}+\frac{c}{2}\|\mathbf{A} \mathbf{x}(q)-\mathbf{z}\|_{2}^{2}\right\}  \tag{20a}\\
& \mathbf{x}(q+1)=\arg \min _{\mathbf{x} \in \mathcal{S}_{1}}\left\{f_{1}(\mathbf{x})+\boldsymbol{\nu}(q)^{T} \mathbf{A} \mathbf{x}+\frac{c}{2}\|\mathbf{A} \mathbf{x}-\mathbf{z}(q+1)\|_{2}^{2}\right\} \tag{20b}
\end{align*}
$$

## ADMM Algorithm

1: Set $q=0$, choose $c>0$.
2: Initialize $\boldsymbol{\nu}(q)$ and $\mathbf{x}(q)$.

## repeat

4: Solve (20a) and (20b) for $\mathbf{z}(q+1)$ and $\mathbf{x}(q+1)$ by two distributed equipments including the information exchange of $\mathbf{z}(q+1)$ and $\mathbf{x}(q+1)$ between them;
5: $\quad \boldsymbol{\nu}(q+1)=\boldsymbol{\nu}(q)+c(\mathbf{A} \mathbf{x}(q+1)-\mathbf{z}(q+1))$;
6: $\quad q:=q+1$;
7: until the predefined stopping criterion is satisfied.

- When $\mathcal{S}_{1}$ is bounded or $\mathbf{A}^{T} \mathbf{A}$ is invertible, ADMM is guaranteed to converge and the obtained $\{\mathbf{x}(q), \mathbf{z}(q)\}$ is an optimal solution of problem (18).


## Nonconvex problem

- Reformulation into a convex problem:

Equivalent representations (e.g. epigraph representations); function transformation; change of variables, etc.

- Stationary-point solutions: Provided that $\mathcal{C}$ is closed and convex but $f$ is nonconvex, a point $\mathbf{x}^{\star}$ is a stationary point of the nonconvex problem (1) if

$$
\begin{align*}
f^{\prime}\left(\mathbf{x}^{\star} ; \mathbf{v}\right) & \triangleq \liminf _{\lambda \downarrow 0} \frac{f\left(\mathbf{x}^{\star}+\lambda \mathbf{v}\right)-f\left(\mathbf{x}^{\star}\right)}{\lambda} \geq 0 \quad \forall \mathbf{x}^{\star}+\mathbf{v} \in \mathcal{C}  \tag{21}\\
& \Leftrightarrow \nabla f\left(\mathbf{x}^{\star}\right)^{T}\left(\mathbf{x}-\mathbf{x}^{\star}\right) \geq 0 \quad \forall \mathbf{x} \in \mathcal{C} \quad \text { (when } f \text { is differentiable) }
\end{align*}
$$

where $f^{\prime}\left(\mathbf{x}^{\star} ; \mathbf{v}\right)$ is the directional derivative of $f$ at a point $\mathbf{x}^{\star}$ in direction $\mathbf{v}$. Block successive upper bound minimization (BSUM) [Razaviyayn'13] guarantees a stationary-point solution under some convergence conditions.

- KKT points (i.e., solutions of KKT conditions) are also stationary points under some mild condition provided that the Slater's condition is satisfied.

[^0]
## Stationary points and BSUM

- An illustration of stationary points of problem (1) for a nonconvex $f$ and convex $\mathcal{C}$; convergence to a stationary point of (1) by BSUM.



## Stationary points for nonconvex feasible set

- An illustration of stationary points of problem (1) when both $f$ and $\mathcal{C}$ are nonconvex. If $\boldsymbol{y}_{1}, \boldsymbol{y}_{2}, \boldsymbol{y}_{3}$ are stationary points of $\min _{\mathbf{x} \in C} f(\mathbf{x})$ where $C \subset \mathcal{C}$ is convex, then conic $\left(C-\left\{\boldsymbol{y}_{i}\right\}\right)=\left\{\theta \mathbf{v} \mid \mathbf{v} \in C-\left\{\boldsymbol{y}_{i}\right\}, \theta \geq 0\right\}$ and

$$
\begin{aligned}
& \mathcal{C}-\left\{\boldsymbol{y}_{i}\right\} \triangleq\left\{\mathbf{v}=\mathbf{x}-\left\{\boldsymbol{y}_{i}\right\} \mid \mathbf{x} \in \mathcal{C}\right\} \subset \mathbf{c o n i c}\left(C-\left\{\boldsymbol{y}_{i}\right\}\right), i=1,2 \\
& \Longrightarrow \boldsymbol{y}_{1}, \boldsymbol{y}_{2} \text { are also stationary points of }(1)
\end{aligned}
$$



## Nonconvex problem

- Approximate solutions to problem (1) when $f$ is convex but $\mathcal{C}$ is nonconvex:
- Convex restriction to $\mathcal{C}$ : Successive convex approximation (SCA)

$$
\begin{equation*}
\boldsymbol{x}_{i}^{\star}=\arg \min _{\mathbf{x} \in C_{i}} f(\mathbf{x}) \in C_{i+1} \tag{22}
\end{equation*}
$$

where $C_{i} \subset \mathcal{C}$ is convex for all $i$. Then

$$
\begin{equation*}
f\left(\boldsymbol{x}_{i+1}^{\star}\right)=\min _{\mathbf{x} \in C_{i+1}} f(\mathbf{x}) \leq f\left(\boldsymbol{x}_{i}^{\star}\right) \tag{23}
\end{equation*}
$$

After convergence, an approximate solution $\boldsymbol{x}_{i}^{\star}$ is obtained.

- Convex relaxation to $\mathcal{C}$ (e.g., semidefinite relaxation (SDR)):

$$
\begin{array}{r}
\mathcal{C}=\left\{\mathbf{X} \in \mathbb{S}_{+}^{n} \mid \operatorname{rank}(\mathbf{X})=1\right\} \text { relaxed to conv } \mathcal{C}=\mathbb{S}_{+}^{n} \quad(\mathrm{SDR}) ; \\
\mathcal{C}=\{-3,-1,+1,+3\} \text { relaxed to conv } \mathcal{C}=[-3,3] \tag{24}
\end{array}
$$

The obtained $\boldsymbol{X}^{\star}$ or $\boldsymbol{x}^{\star}$ may not be feasible to problem (1); for SDR, a good approximate solution can be obtained from $\boldsymbol{X}^{\star}$ via Gaussian randomization.

## Successive Convex Approximation (SCA)

- Illustration of SCA for (1) when $f$ is convex but $\mathcal{C}$ is nonconvex, yielding a solution $\mathbf{x}_{i}^{\star}$ (which is a stationary point under some mild condition).



## Algorithm development

- Foundamental theory and tools: Calculus, linear algebra, matrix analysis and computations, convex sets, convex functions, convex problems (e.g., geometric program (GP), LP, QP, SOCP, SDP), duality, interior-point method; CVX and SeDuMi.



## A new book

Convex Optimization for Signal Processing and Communications: From Fundamentals to Applications
Chong-Yung Chi, Wei-Chiang Li, Chia-Hsiang Lin (Publisher: CRC Press, 2017, 432 pages, ISBN: 9781498776455)

Motivation: Most of mathematical books are hard to read for engineering students and professionals due to lack of enough fundamental details and tangible linkage between mathematical theory and applications.

- The book is written in a causally sequential fashion; namely, one can review/peruse the related materials introduced in early chapters/sections again, to overpass hurdles in reading.
- Covers convex optimization from fundamentals to advanced applications, while holding a strong link from theory to applications.
- Provides comprehensive proofs and perspective interpretations, many insightful figures, examples and remarks to illuminate core convex optimization theory.


## Book features



From Fundamentals to Applications

Chong-Yung-Chi - Wei-Chiang Li - Chia-Hsiang Lin
${ }^{C B^{C}}$ CRC Press

- Illustrates, by cutting-edge applications, how to apply the convex optimization theory, like a guided journey/exploration rather than pure mathematics.
- Has been used for a 2-week short course under the book title at 9 major universities (Shandong Univ, Tsinghua Univ, Tianjin Univ, BJTU, Xiamen Univ., UESTC, SYSU, BUPT, SDNU) in Mainland China more than 17 times since early 2010.

Thank you for your attention!

Acknowledgment: Financial support by NTHU; my students, visiting students and scholars, short-course participants for almost uncountable questions, interactions and comments over the last decade.

## Outline

(1) Part I: Fundamentals of Convex Optimization
(2) Part II: Application in Hyperspectral Image Analysis: (Big Data Analysis and Machine Learning)
(3) Part III: Application in Wireless Communications (5G Systems)

- Subsection I: Outage Constrained Robust Transmit Optimization for Multiuser MISO Downlinks
- Subsection II: Outage Constrained Robust Hybrid Coordinated Beamforming for Massive MIMO Enabled Heterogeneous Cellular Networks


## Introduction to hyperspectral unmixing (HU)

A hyperspectral sensor records electromagnetic 'fingerprints' (scattering patterns) of substances (materials) in a scene, known as spectral signatures, over hundreds of spectral bands from visible to short-wave infrared wavelength region.


Issue: mixed pixel spectra in the hyperspectral image/data (due to limited spatial resolution of the hypersepctral sensor) must be decomposed for identifying the underlying materials (i.e., spectral unmixing) [Keshava'02].

Airborne Sensors. Courtesy: http://masterweb.jpl.nasa.gov/

## Introduction to hyperspectral unmixing (HU)



Airborne Sensors. Courtesy: http://masterweb.jpl.nasa.gov/

Hyperspectral unmixing (HU) is a signal processing procedure of extracting hidden spectral signatures of substances (called endmembers) and the corresponding proportions (called abundances) (i.e., distribution map of substances) in a scene, from the hyperspectral observations [Keshava'02].

Applications: Terrain classification, mineral identification and quantification, agricultural monitoring, military surveillance, space object identification, etc.

[^1]
## Illustration of HU [Ambikapathi'11]

- Red pixel: a mixed pixel (land+vegetation+water)
- Blue pixel: a pure pixel (only water)

[Ambikapathi'11] A. Ambikapathi et al., "Chance constrained robust minimum volume enclosing simplex algorithm for hyperspectral unmixing," IEEE Trans. Geoscience and Remote Sensing, vol. 49, no. 11, pp. 4194-4209, Nov. 2011.


## Signal model and assumptions

## Linear Mixing Model

The pixel vector $\mathbf{x}[n]$ of $M$ spectral bands can be represented as a linear combination of $N$ endmember signatures $\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}\right\}$, i.e.,

$$
\mathbf{x}[n]=\left\{\begin{array}{l}
\mathbf{A s}[n]=\sum_{i=1}^{N} s_{i}[n] \mathbf{a}_{i} \in \mathbb{R}^{M}, n \in \mathcal{I}_{L} \triangleq\{1, \ldots, L\} .  \tag{25}\\
s_{j}[n] \mathbf{a}_{j} \text { (a pure pixel) if } s_{i}[n]=0 \forall i \neq j
\end{array}\right.
$$

- $\mathbf{A}=\left[\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}\right] \in \mathbb{R}^{M \times N}$ (spectral signature matrix or mixing matrix).
- $\mathbf{s}[n] \triangleq\left[s_{1}[n], \ldots, s_{N}[n]\right]^{T} \in \mathbb{R}^{N}$ is the $n$th abundance vector; $\boldsymbol{s}_{i}=\left\{s_{i}[n] \mid n \in \mathcal{I}_{L}\right\}$ is called source $i$ or abundance map $i$.


## Standard Assumptions (Non-statistical) [Keshava'02]

(A1) $\mathbf{s}[n] \succeq \mathbf{0}_{N}$ and $\sum_{i=1}^{N} s_{i}[n]=1, \forall n \in \mathcal{I}_{L}$.
(A2) $\mathbf{A}=\left[\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}\right] \succeq \mathbf{0}_{M \times N}$ has full column rank and $\min \{L, M\} \geq N$.

## Craig's HU criterion

- Observation: $\boldsymbol{X} \triangleq\{\mathbf{x}[1], \ldots, \mathbf{x}[L]\} \subset \operatorname{conv}\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}\right\} \triangleq \mathcal{T}_{a} \subset \mathbb{R}^{M}$ (an $N$-vertex simplex due to (A1) and (A2)); conv $\boldsymbol{X}=\mathcal{T}_{a}$ if $\underline{\mathbf{a}_{i} \in \boldsymbol{X} \forall i}$.
- Craig's belief: The vertices of the minimum-volume data-enclosing simplex $\widehat{\mathcal{T}}_{a}$ yield good estimate $\widehat{\mathbf{a}}_{i}$ [Craig'94] even without any pure pixels, $\mathbf{a}_{i}$ in $\boldsymbol{X}$.


Figure 2: $\operatorname{Visually}, \operatorname{vol}\left(\mathcal{T}_{a}\right)<\operatorname{vol}\left(\mathcal{T}_{i}\right), i \in \mathcal{I}_{2}$ (the dots are data points $\left.\mathbf{x}[n]\right)$.

## Craig's HU criterion (Cont'd)

- Craig's criterion [Craig'94] (an NP-hard problem):

$$
\begin{array}{r}
\left\{\widehat{\mathbf{a}}_{1}, \ldots, \widehat{\mathbf{a}}_{N}\right\} \in \arg \min _{\boldsymbol{b}_{i} \in \mathbb{R}^{M} \forall i} \operatorname{vol}\left(\operatorname{conv}\left\{\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{N}\right\}\right)  \tag{26}\\
\text { s.t. } \boldsymbol{X} \subset \operatorname{conv}\left\{\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{N}\right\}
\end{array}
$$

- In [Lin'15], we theoretically proved that as long as the data uniform purity level $\gamma$ is above a threshold (a mild condition), i.e.,

$$
\gamma \triangleq \max \left\{r \mid \mathcal{T}_{e} \cap \mathcal{B}(r) \subseteq \operatorname{conv}\{\mathbf{s}[1], \ldots, \mathbf{s}[L]\}>1 / \sqrt{N-1}\right.
$$

where $\mathcal{T}_{e} \triangleq \operatorname{conv}\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{N}\right\} \subseteq \mathbb{R}^{N}$ (unit simplex) and $\mathcal{B}(r) \triangleq\left\{\boldsymbol{x} \in \mathbb{R}^{N} \mid\|\boldsymbol{x}\| \leq r\right\}$, Craig's criterion can perfectly identify the ground-truth endmembers $\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{N}\right\}$ (i.e., the true simplex $\mathcal{T}_{a}$ ).

- Can we devise a super-efficient HU algorithm using Craig's criterion?
[Craig'94] M. D. Craig, "Minimum-volume transforms for remotely sensed data," IEEE Trans. Geosci. Remote Sens., vol. 32, no. 3, pp. 542-552, May 1994.
[Lin'15] C.-H. Lin et al., "Identifiability of the simplex volume minimization criterion for blind hyperspectral unmixing: The no pure-pixel case," IEEE Trans. Geosci. Remote Sens., vol. 53, no.10, pp. 5530-5546, Oct. 2015.


## Dimension reduction and problem formulation

- Dimension-reduced (DR) representation for noise suppression and computational complexity reduction: Due to affdim $\mathcal{T}_{a}=N-1$,

$$
\begin{align*}
\mathbf{x}[n] & =\mathbf{C} \widetilde{\mathbf{x}}[n]+\mathbf{d} \in \underline{\boldsymbol{X} \subset \mathcal{T}_{a} \subset \text { aff } \mathcal{T}_{a} \subset \mathbb{R}^{M}} \text { (affine set fitting) }  \tag{27}\\
\Rightarrow \widetilde{\mathbf{x}}[n] & =\mathbf{C}^{\dagger}(\mathbf{x}[n]-\mathbf{d}) \\
& =\sum_{i=1}^{N} s_{i}[n] \boldsymbol{\alpha}_{i} \in \mathcal{T}_{\alpha}=\operatorname{conv}\left\{\boldsymbol{\alpha}_{1}, \ldots, \boldsymbol{\alpha}_{N}\right\} \subset \text { aff } \mathcal{T}_{\alpha} \subset \mathbb{R}^{N-1} \tag{28}
\end{align*}
$$

- $\mathbf{C} \triangleq\left[\boldsymbol{q}_{1}\left(\mathbf{U U}^{T}\right), \ldots, \boldsymbol{q}_{N-1}\left(\mathbf{U U}^{T}\right)\right]$ (a semiunitary matrix), where $\boldsymbol{q}_{i}(\cdot)$ is the $i$ th principal eigenvector, and $\mathbf{U} \triangleq[\mathbf{x}[1]-\mathbf{d}, \ldots, \mathbf{x}[L]-\mathbf{d}] \in \mathbb{R}^{M \times L}$ (mean removed data matrix);
- $\mathbf{d} \triangleq \frac{1}{L} \sum_{n=1}^{L} \mathbf{x}[n]$ (mean of the data set $\boldsymbol{X}$ );
- $\mathcal{T}_{\alpha}$ is an $N$-vertex simplest simplex;

$$
\begin{align*}
\mathcal{X} & =\{\widetilde{\mathbf{x}}[1], \ldots, \widetilde{\mathbf{x}}[L]\} \subset \mathcal{T}_{\alpha} \subset \mathbb{R}^{N-1} \quad(D R \text { data set })  \tag{29}\\
\boldsymbol{\alpha}_{i} & \triangleq \mathbf{C}^{\dagger}\left(\mathbf{a}_{i}-\mathbf{d}\right) \in \mathbb{R}^{N-1}(D R \text { endmembers }) \tag{30}
\end{align*}
$$

## Illustration of Dimension Reduction



- Dimension reduction illustration using affine set fitting for $N=3$, where the geometric center $\mathbf{d}$ of the data cloud $\mathcal{X}$ in the $M$-dimensional space maps to the origin in the $(N-1)$-dimensional space.


## Dimension reduction and problem formulation

- Problem (26) can be reformulated in the DR space as:

$$
\begin{gather*}
\left\{\widehat{\boldsymbol{\alpha}}_{1}, \ldots, \widehat{\boldsymbol{\alpha}}_{N}\right\} \in \arg \min _{\boldsymbol{\beta}_{i} \in \mathbb{R}^{N-1} \forall i}\left\{\operatorname{vol}\left(\operatorname{conv}\left\{\boldsymbol{\beta}_{1}, \ldots, \boldsymbol{\beta}_{N}\right\}\right)=\frac{|\operatorname{det}(\mathbf{B})|}{(N-1)!}\right\} \\
\text { s.t. } \mathcal{X} \subset \operatorname{conv}\left\{\boldsymbol{\beta}_{1}, \ldots, \boldsymbol{\beta}_{N}\right\} \tag{31}
\end{gather*}
$$

where $\mathbf{B}=\left[\boldsymbol{\beta}_{1}-\boldsymbol{\beta}_{N}, \ldots, \boldsymbol{\beta}_{N-1}-\boldsymbol{\beta}_{N}\right] \in \mathbb{R}^{(N-1) \times(N-1)}$.

- Endmember estimates in the original space $\mathbb{R}^{M}$ :

$$
\widehat{\mathbf{a}}_{i}=\mathbf{C} \widehat{\boldsymbol{\alpha}}_{i}+\mathbf{d}, \forall i \in \mathcal{I}_{N} \quad \text { (cf. (30)) }
$$

## Existing Challenges

- Pure pixel assumption (PPA) enables various simple and fast blind HU algorithmic schemes (for finding the purest pixels in the data set $\boldsymbol{X}$ or the $D R$ data set $\mathcal{X}$ ), but it is often seriously infringed.
- Without requiring the PPA, Craig's blind HU criterion identifies the $N$-vertex minimum-volume data-enclosing simplex $\widehat{\mathcal{T}}_{a} \subset \mathbb{R}^{M}$, but suffering from heavy simplex volume computations.


## New Philosophy: Hyperplane-based CSI Algorithm

## Geometry Fact 1 [Lin'16]:

$$
\underline{s_{i}[n]=0} \text { if and only if } \widetilde{\mathbf{x}}[n] \in \mathcal{H}_{i},
$$

where $\mathcal{H}_{i} \triangleq \operatorname{aff}\left(\left\{\boldsymbol{\alpha}_{1}, \ldots, \boldsymbol{\alpha}_{N}\right\} \backslash\left\{\boldsymbol{\alpha}_{i}\right\}\right) \subset \mathbb{R}^{N-1}$ is the boundary hyperplane of the DR endmembers' simplest simplex $\mathcal{T}_{\alpha} \subset \mathbb{R}^{N-1}$.

## Observation

The abundance maps often show large sparseness [Cichocki'09], and many pixels lying on or close to the boundary hyperplanes of the Craig's simplex.

## Geometry Fact 2:

A simplest simplex of $N$ vertices can be defined by its $N$ boundary hyperplanes $\mathcal{H}_{i}$ (each containing $N-1$ vertices) and vice versa.

## Real data experiments



Muscovite


Hematite


Kaolinite \#2


Alunite


Montmorillonite


Buddingtonite


Desert varnish


Kaolinite \#1


Chalcedony

The good performance of HyperCSI in the experiment also implies that the requirement of sufficient (i.e., $N(N-1)=72$ ) active pixels lying close to the hyperplanes of the actual endmembers' simplex, has been met for the considered hyperspectral scene.

## Super-fast HU algorithm: HyperCSI

## HyperCSI Algorithm: A novel practical HU algorithm

- HyperCSI algorithm tries to estimate the $N$ boundary hyperplanes of the $N$-vertex simplest simplex $\widehat{\mathcal{T}}_{\alpha} \subset \mathbb{R}^{N-1}$ (in DR space), without any simplex volume computations.
- Each hyperplane is constructed by $N-1$ A.I. data pixels that are identified via simple linear algebraic computations.
- With better performance than state-of-the-art algorithms, HyperCSI is at least computationally efficient as PPA based HU algorithms.

[^2]
## Block diagram of HyperCSI algorithm

- An algorithm with parallel processing structure for estimation of normal vectors ( $\mathbf{b}_{i}$ ), inner product parameters ( $h_{i}$ ), and abundance maps ( $\boldsymbol{s}_{i}$ ), where the PPA based successive projection algorithm (SPA) is employed to obtain initial estimates $\widetilde{\alpha}_{1}, \ldots, \widetilde{\alpha}_{N}$.



## Graphical illustration of HyperCSI



- Why $\mathcal{R}_{k}^{(i)}$ should be disjoint? Consider $\left\{\mathbf{p}_{2}^{(1)}, \mathbf{q}\right\}$ identified by (34).
- Why not $\widetilde{\mathbf{b}}_{1}$ ? The purest pixel $\widetilde{\boldsymbol{\alpha}}_{3}$ may not be close to $\mathcal{H}_{1}=\operatorname{aff}\left\{\boldsymbol{\alpha}_{2}, \boldsymbol{\alpha}_{3}\right\}$, leading to nontrivial orientation difference between $\widetilde{\mathbf{b}}_{1}$ and $\mathbf{b}_{1}$.
- However, $\left\{\mathbf{p}_{1}^{(1)}, \mathbf{p}_{2}^{(1)}\right\}$ identified by (35) are very close to $\mathcal{H}_{1}$, so the orientations of $\widehat{\mathbf{b}}_{1}$ and $\mathbf{b}_{1}$ are almost the same.


## HyperCSI Algorithm

## Pseudocode for the HyperCSI Algorithm

(1) Given Hyperspectral data $\{\mathbf{x}[1], \ldots, \mathbf{x}[L]\}$, number of endmembers $N$, and $\eta=0.9$.
(2) Obtain DR dataset $\mathcal{X}=\{\widetilde{\mathbf{x}}[1], \ldots, \widetilde{\mathbf{x}}[L]\}$ by (28). (DR processing)
(3) Obtain purest pixels $\left\{\widetilde{\boldsymbol{\alpha}}_{1}, \ldots, \widetilde{\boldsymbol{\alpha}}_{N}\right\}$ by SPA. (preprocessing)
(4) Obtain $\widetilde{\mathbf{b}}_{i}=\boldsymbol{v}_{i}\left(\widetilde{\boldsymbol{\alpha}}_{1}, \ldots, \widetilde{\boldsymbol{\alpha}}_{N}\right)$ by (32), and then obtain A.I. set $\left\{\mathbf{p}_{1}^{(i)}, \ldots, \mathbf{p}_{N-1}^{(i)}\right\}$ by (35) $\forall i \in \mathcal{I}_{N}$.
(5) Obtain $\widehat{\mathbf{b}}_{i}$ and $\widehat{h}_{i}$ by (36) and (37), respectively, $\forall i \in \mathcal{I}_{N}$.
(6) Obtain $c^{\prime}$ by (38), and set $c=c^{\prime} / \eta$. (further denoising)
(7) Obtain $\widehat{\boldsymbol{\alpha}}_{i}$ by (39) and then obtain $\widehat{\mathbf{a}}_{i}$ by (40) $\forall i \in \mathcal{I}_{N}$.(identifiable w.p.1)
(8) Calculate $\widehat{\mathbf{s}}[n]=\left[\widehat{s}_{1}[n], \ldots, \widehat{s}_{N}[n]\right]^{T}$ by (42) for all $n \in \mathcal{I}_{L}$. $\mathcal{O}\left(N^{2} L\right)$
(9) Output The endmember estimates $\left\{\widehat{\mathbf{a}}_{1}, \ldots, \widehat{\mathbf{a}}_{N}\right\}$ and abundance maps $\left\{\widehat{\boldsymbol{s}}_{1}, \ldots, \widehat{\boldsymbol{s}}_{N}\right\}$.

## Hyperplane representation of Simplest Simplex

- Since $\mathcal{T}_{\alpha}$ is a simplest simplex in the DR space $\mathbb{R}^{N-1}$, enabling the following handy parameterization for $\mathcal{H}_{i}$ :

$$
\mathcal{H}_{i} \triangleq \operatorname{aff}\left(\mathcal{T}_{\alpha} \backslash\left\{\boldsymbol{\alpha}_{i}\right\}\right)=\left\{\mathbf{x} \in \mathbb{R}^{N-1} \mid \mathbf{b}_{i}^{T} \mathbf{x}=h_{i}\right\} \equiv \mathcal{H}\left(\mathbf{b}_{i}, h_{i}\right)
$$

where $\mathbf{b}_{i} \in \mathbb{R}^{N-1}$ and $h_{i} \in \mathbb{R}$ respectively denote the outward-pointing normal vector of $\mathcal{H}_{i}$ and the inner product constant of $\mathcal{H}_{i}$.

## Proposition 1 [Lin'16]

The endmembers $\boldsymbol{\alpha}_{i}$ can then be reconstructed by the hyperplanes $\mathcal{H}_{i}$ :

$$
\boldsymbol{\alpha}_{i}=\mathbf{B}_{-i}^{-1} \mathbf{h}_{-i}, \forall i \in \mathcal{I}_{N}
$$

where $\mathbf{B}_{-i} \triangleq\left[\mathbf{b}_{1}, \ldots, \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_{N}\right]^{T}$;
$\mathbf{h}_{-i} \triangleq\left[h_{1}, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{N}\right]^{T}$.

- Problem (31) can be decoupled into $N$ parallel subproblems of estimating $\left(\mathbf{b}_{i}, h_{i}\right), \forall i \in \mathcal{I}_{N}$. Next, let us focus on how to estimate $\left(\mathbf{b}_{i}, h_{i}\right), \forall i \in \mathcal{I}_{N}$.


## Normal vector $\mathrm{b}_{i}$ estimation

- The normal vector $\mathbf{b}_{i}$ can be reconstructed by $N-1 \mathcal{A}$.I. points on $\mathcal{H}_{i}$.


## Proposition 2 [Lin'16]

Given A.I. $\left\{\mathbf{p}_{1}^{(i)}, \ldots, \mathbf{p}_{N-1}^{(i)}\right\} \subseteq \mathcal{H}_{i}$, we have

$$
\begin{align*}
\mathbf{b}_{i} & =\boldsymbol{v}_{i}\left(\mathbf{p}_{1}^{(i)}, \ldots, \mathbf{p}_{i-1}^{(i)}, \mathbf{0}_{N-1}, \mathbf{p}_{i}^{(i)}, \ldots, \mathbf{p}_{N-1}^{(i)}\right)  \tag{32}\\
& \triangleq\left(\mathbf{I}_{N-1}-\mathbf{P}\left(\mathbf{P}^{T} \mathbf{P}\right)^{-1} \mathbf{P}^{T}\right) \cdot \mathbf{p}_{k}^{(i)}
\end{align*}
$$

for arbitrary $k \in \mathcal{I}_{N-1}$, where

$$
\begin{aligned}
& \mathbf{P} \triangleq \mathbf{Q}-\mathbf{p}_{k}^{(i)} \cdot \mathbf{1}_{N-2}^{T} \in \mathbb{R}^{(N-1) \times(N-2)}, \\
& \mathbf{Q} \triangleq\left[\mathbf{p}_{1}^{(i)}, \ldots, \mathbf{p}_{k-1}^{(i)}, \mathbf{p}_{k+1}^{(i)}, \ldots, \mathbf{p}_{N-1}^{(i)}\right] \in \mathbb{R}^{(N-1) \times(N-2)} .
\end{aligned}
$$

- Motivated by Proposition 2, we aim at extracting $N-1 \mathcal{A} . \mathcal{I}$. data pixels $\mathbf{p}_{k}^{(i)}$ from the DR dataset $\mathcal{X}$ that are closest to $\mathcal{H}_{i}$.


## Normal vector $\mathrm{b}_{i}$ Estimation (Cont’d)

## Proposition 3 [Lin'16]

Observing that all the points $\mathbf{p} \in \mathcal{X}$ lie on the same side of $\mathcal{H}_{i}$, i.e.,

$$
\mathbf{b}_{i}^{T} \mathbf{p} \leq h_{i}, \forall \mathbf{p} \in \mathcal{X}, \text { and } \operatorname{dist}\left(\mathbf{p}, \mathcal{H}_{i}\right)=\left|h_{i}-\mathbf{b}_{i}^{T} \mathbf{p}\right| /\left\|\mathbf{b}_{i}\right\|,
$$

we see that the point $\mathbf{p} \in \mathcal{X}$ closest to $\mathcal{H}_{i}$ is exactly the one with maximum of $\mathbf{b}_{i}^{T} \mathbf{p}$, because $\mathbf{b}_{i}$ is outward-pointing.

- Let $\widetilde{\mathcal{H}}_{i} \triangleq \operatorname{aff}\left(\left\{\widetilde{\boldsymbol{\alpha}}_{1}, \ldots, \widetilde{\boldsymbol{\alpha}}_{N}\right\} \backslash\left\{\widetilde{\boldsymbol{\alpha}}_{i}\right\}\right) \equiv \mathcal{H}\left(\widetilde{\mathbf{b}}_{i}, \widetilde{h}_{i}\right)$, where $\widetilde{\boldsymbol{\alpha}}_{i}$ are the purest pixels extracted by the successive projection algorithm [Arora'12].
- By Proposition 2 and Proposition 3, a naive way for estimating $\mathbf{p}_{k}^{(i)}$ is

$$
\begin{equation*}
\left\{\mathbf{p}_{1}^{(i)}, \ldots, \mathbf{p}_{N-1}^{(i)}\right\} \in \arg \max \left\{\widetilde{\mathbf{b}}_{i}^{T}\left(\mathbf{p}_{1}+\cdots+\mathbf{p}_{N-1}\right) \mid \mathbf{p}_{k} \in \mathcal{X}\right\} \tag{33}
\end{equation*}
$$

$\Longrightarrow$ The identified $\mathbf{p}_{k}^{(i)}$ can be quite close to each other.
[Arora'12] S. Arora et al., "A practical algorithm for topic modeling with provable guarantees," arXiv preprint arXiv:1212.4777, 2012.

## Normal vector $\mathrm{b}_{i}$ Estimation (Cont'd.)



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## Normal vector $\mathrm{b}_{i}$ Estimation (Cont'd)

- Accordingly, $\mathbf{p}_{k}^{(i)}$ for $\mathcal{H}_{i}$ are sifted by

$$
\begin{equation*}
\mathbf{p}_{k}^{(i)} \in \arg \max \left\{\widetilde{\mathbf{b}}_{i}^{T} \mathbf{p} \mid \mathbf{p} \in \mathcal{X} \cap \mathcal{R}_{k}^{(i)}\right\}, \forall k \in \mathcal{I}_{N-1} \tag{35}
\end{equation*}
$$

where $\mathcal{R}_{k}^{(i)}$ are disjoint regions (norm balls with the same radius) defined by

$$
\begin{aligned}
\mathcal{R}_{k}^{(i)} & \triangleq\left\{\begin{array}{l}
\mathcal{B}\left(\widetilde{\boldsymbol{\alpha}}_{k}, r\right), \text { if } k<i, \\
\mathcal{B}\left(\widetilde{\boldsymbol{\alpha}}_{k+1}, r\right), \text { if } k \geq i,
\end{array}\right. \\
\mathcal{B}\left(\widetilde{\boldsymbol{\alpha}}_{k}, r\right) & \triangleq\left\{\mathbf{x} \in \mathbb{R}^{N} \mid\left\|\mathbf{x}-\widetilde{\boldsymbol{\alpha}}_{k}\right\|<r\right\}, \\
r & \triangleq(1 / 2) \cdot \min \left\{\left\|\widetilde{\boldsymbol{\alpha}}_{i}-\widetilde{\boldsymbol{\alpha}}_{j}\right\| \mid 1 \leq i<j \leq N\right\}>0 .
\end{aligned}
$$

- Normal vector estimates:

$$
\begin{equation*}
\widehat{\mathbf{b}}_{i}=\boldsymbol{v}_{i}\left(\mathbf{p}_{1}^{(i)}, \ldots, \mathbf{p}_{i-1}^{(i)}, \mathbf{0}_{N-1}, \mathbf{p}_{i}^{(i)}, \ldots, \mathbf{p}_{N-1}^{(i)}\right) \quad \forall i \tag{36}
\end{equation*}
$$

## Inner product constant $h_{i}$ Estimation

- Craig's simplex must tightly enclose $\mathcal{X}$ :

$$
\begin{equation*}
\widehat{h}_{i}=\max \left\{\widehat{\mathbf{b}}_{i}^{T} \mathbf{p} \mid \mathbf{p} \in \mathcal{X}\right\} . \tag{37}
\end{equation*}
$$

- Considering the simplex volume expansion caused by noise, the estimated hyperplanes need to be properly shifted closer to the origin, or equivalently $\widehat{h}_{i}$ should be scaled down as $\widehat{h}_{i} / c$ for some $c \geq 1$ in general.

Non-data dependent choice of the parameter $c$
As $\mathbf{A} \succeq \mathbf{0}_{M \times N}$, Proposition 1 indicates $c \geq c^{\prime}$ (empirically, $c=c^{\prime} / \eta$ and $\eta=0.9)$, where

$$
\begin{align*}
c^{\prime} & \triangleq \min _{c^{\prime \prime} \geq 1}\left\{c^{\prime \prime} \mid \mathbf{C}\left(\widehat{\mathbf{B}}_{-i}^{-1} \cdot \hat{\mathbf{h}}_{-i}\right)+c^{\prime \prime} \cdot \mathbf{d} \succeq \mathbf{0}_{M}, \forall i \in \mathcal{I}_{N}\right\} \\
& =\max \left\{1, \max \left\{-v_{i j} / d_{j} \mid i \in \mathcal{I}_{N}, j \in \mathcal{I}_{M}\right\}\right\} \tag{38}
\end{align*}
$$

in which $v_{i j}$ is the $j$ th entry of $\mathbf{C}\left(\widehat{\mathbf{B}}_{-i}^{-1} \cdot \widehat{\mathbf{h}}_{-i}\right)$ and $d_{j}$ is the $j$ th entry of $\mathbf{d}$.

- Endmember estimates:

$$
\begin{align*}
& \widehat{\boldsymbol{\alpha}}_{i}=\widehat{\mathbf{B}}_{-i}^{-1} \widehat{\mathbf{h}}_{-i} / c \in \mathbb{R}^{N-1},  \tag{39}\\
& \widehat{\mathbf{a}}_{i}=\mathbf{C} \widehat{\boldsymbol{\alpha}}_{i}+\mathbf{d} \succeq \mathbf{0}_{M}, \forall i \in \mathcal{I}_{N} . \tag{40}
\end{align*}
$$

## Abundance vector $\mathrm{s}[n]$ estimation

## Lemma 3 [Lin'16] (Closed-form Expression of $\mathbf{s}[n]$ )

Assume that (A1) and (A2) hold true. Then, $\mathbf{s}[n]=\left[s_{1}[n], \ldots, s_{N}[n]\right]^{T}$ has the following closed-form expression:

$$
\begin{equation*}
s_{i}[n]=\frac{h_{i}-\mathbf{b}_{i}^{T} \tilde{\mathbf{x}}[n]}{h_{i}-\mathbf{b}_{i}^{T} \boldsymbol{\alpha}_{i}}, \quad \forall i \in \mathcal{I}_{N}, \forall n \in \mathcal{I}_{L} \tag{41}
\end{equation*}
$$

$$
\begin{equation*}
\Rightarrow \widehat{s}_{i}[n]=\max \left\{\frac{\widehat{h}_{i}-\widehat{\mathbf{b}}_{i}^{T} \tilde{\mathbf{x}}[n]}{\widehat{h}_{i}-\widehat{\mathbf{b}}_{i}^{T} \widehat{\boldsymbol{\alpha}}_{i}}, 0\right\}, \quad \forall i \in \mathcal{I}_{N}, \forall n \in \mathcal{I}_{L} \tag{42}
\end{equation*}
$$

## Theorem 1 (Computational Complexity Analysis) [Lin'16]

The computational complexity of HyperCSI is $\mathcal{O}(N L)$ with parallel processing and $\mathcal{O}\left(N L^{2}\right)$ without parallel processing.

## Statistical analysis

(A3) $\mathrm{s}[1], \ldots, \mathrm{s}[L]$ are independent and identically distributed (i.i.d.) with a continuous probability density function (p.d.f.) whose support is the unit simplex $\mathcal{T}_{e} \triangleq \operatorname{conv}\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{N}\right\} \subseteq \mathbb{R}^{N}$ (e.g., Dirichlet distribution).

## Theorem 2 (Affine Independence Analysis) [Lin'16]

Under (A1)-(A3), the pixels $\left\{\widehat{\mathbf{p}}_{1}^{(i)}, \ldots, \widehat{\mathbf{p}}_{N-1}^{(i)}\right\}$ extracted by (35) are $\mathcal{A} . \mathcal{I}$. with probability 1 (w.p.1), $\forall i \in \mathcal{I}_{N}$.

## Theorem 3 (Identifiability of HyperCSI Algorithm) [Lin'16]

Under (A1)-(A3), the noiseless assumption and $L \rightarrow \infty$, the simplex identified by the HyperCSI algorithm with $c=1$ is exactly the Craig's simplex and the true endmembers' simplex $\operatorname{conv}\left\{\boldsymbol{\alpha}_{1}, \ldots, \boldsymbol{\alpha}_{N}\right\}$ in the DR space w.p.1.

## Numerical simulaitons

- HyperCSI is compared with five state-of-the-art algorithms [Ma'14]:
(1) Minimum-volume simplex analysis (MVSA) algorithm;
(2) Interior-point method based MVSA (ipMVSA) algorithm [Li'15];
(3) Minimum-volume enclosing simplex (MVES) algorithm [Chan'09];
(9) Simplex identification via split augmented Lagrangian (SISAL) algorithm;
(5) Minimum-volume constrained non-negative matrix factorization (MVC-NMF) algorithm [Miao'07].

[^3]
## Numerical simulaitons (con't)

- Data generation: $N=6$ endmembers with $M=224$ spectral bands are randomly selected from the US Geological Survey (USGS) library to generate $L=10,000$ noiseless synthetic data, and then added by Gaussian noise.
- Performance measures:
(1) Computation time $T$;
(2) Root-mean-square (RMS) spectral angle error $\phi_{\text {en }}$ (between $\mathbf{a}_{i}$ and $\widehat{\mathbf{a}}_{i}$ ):

$$
\phi_{e n}=\min _{\boldsymbol{\pi} \in \Pi_{N}} \sqrt{\frac{1}{N} \sum_{i=1}^{N}\left[\arccos \left(\frac{\mathbf{a}_{i}^{T} \hat{\mathbf{a}}_{\pi_{i}}}{\left\|\mathbf{a}_{i}\right\| \cdot\left\|\hat{\mathbf{a}}_{\pi_{i}}\right\|}\right)\right]^{2}}
$$

where $\Pi_{N}$ is the set of all the permutations of $\{1, \ldots, N\}$.
(3) RMS angle error $\phi_{a b}$ (between $\boldsymbol{s}_{i}$ and $\widehat{\boldsymbol{s}}_{i}$ ):

$$
\phi_{a b}=\min _{\boldsymbol{\pi} \in \Pi_{N}} \sqrt{\frac{1}{N} \sum_{i=1}^{N}\left[\arccos \left(\frac{\boldsymbol{s}_{i}^{T} \hat{\boldsymbol{s}}_{\pi_{i}}}{\left\|\boldsymbol{s}_{i}\right\| \cdot\left\|\hat{\boldsymbol{s}}_{\pi_{i}}\right\|}\right)\right]^{2}}
$$

## Numerical simulaitons (con't)

- The asymptotic analysis in Theorem 3 (noiseless and $c=1$ ) is illustrated below:

- HyperCSI performs well even with a moderately finite number of pixels $L$, i.e., several tens of thousands, which is typical in HU.
- The curve for larger $N$ converges at a slower rate as we need more pixels (i.e., $N(N-1)$ ) lying close to the boundary hyperplanes for larger $N$.


## Numerical simulaitons (con't)


(a) Ground truth abundance maps of SYNl

(b) Ground truth abundance maps of SYN2

Two sets of sparsely, non-i.i.d. and non-Dirichlet distributed maps are used to generate two synthetic datasets [lordache'12], denoted by SYN1 and SYN2, for performance evaluation, where SYN1 contains $L=10,000$ pixels and SYN2 contains $L=16,900$ pixels, resp. [lordache'12].
[lordache'12] M.-D. lordache et al., "Total variation spatial regularization for sparse hyperspectral unmixing," IEEE Trans. Geosci. Remote Sens., vol. 50, no. 11, pp. 4484-4502, 2012.

## Numerical simulaitons (con't)

- Each simulation result for different SNRs, is obtained by averaging over 100 realizations.

|  | Methods | $\phi_{e n}$ (degrees) |  |  |  |  | $\phi_{a b}$ (degrees) |  |  |  |  | $T$ (seconds) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | SNR (dB) |  |  |  |  | SNR (dB) |  |  |  |  |  |
|  |  | 20 | 25 | 30 | 35 | 40 | 20 | 25 | 30 | 35 | 40 |  |
| SYNI | MVC-NMF | 3.23 | 1.97 | 1.05 | 0.55 | 0.25 | 13.87 | 8.51 | 4.79 | 2.65 | 1.34 | $1.74 \mathrm{E}+2$ |
|  | MVSA | 10.65 | 6.12 | 3.38 | 1.88 | 1.05 | 22.93 | 15.13 | 9.34 | 5.52 | 3.19 | $3.53 \mathrm{E}+0$ |
|  | MVES | 9.55 | 5.49 | 3.60 | 1.96 | 1.22 | 23.89 | 17.35 | 14.49 | 7.78 | 5.66 | $3.42 \mathrm{E}+1$ |
|  | SISAL | 4.43 | 2.89 | 1.81 | 1.18 | 0.86 | 15.85 | 10.39 | 6.89 | 5.29 | 4.65 | $2.66 \mathrm{E}+0$ |
|  | ipMVSA | 11.62 | 6.82 | 3.38 | 2.01 | 1.05 | 24.05 | 16.28 | 9.34 | 5.98 | 3.19 | $1.65 \mathrm{E}+0$ |
|  | HyperCSI | 1.55 | 1.22 | 0.79 | 0.52 | 0.35 | 12.03 | 6.92 | 4.16 | 2.49 | 1.46 | $5.56 \mathrm{E}-2$ |
| SYN2 | MVC-NMF | 2.86 | 1.71 | 0.97 | 0.54 | 0.23 | 22.86 | 15.52 | 9.39 | 5.27 | 2.67 | $2.48 \mathrm{E}+2$ |
|  | MVSA | 10.21 | 5.55 | 3.08 | 1.71 | 0.95 | 29.86 | 22.72 | 15.57 | 9.78 | 5.83 | $5.65 \mathrm{E}+0$ |
|  | MVES | 10.12 | 5.19 | 3.15 | 2.04 | 3.77 | 29.43 | 22.13 | 15.66 | 10.42 | 13.17 | $2.22 \mathrm{E}+1$ |
|  | SISAL | 3.25 | 2.18 | 1.48 | 0.96 | 0.63 | 24.79 | 17.49 | 11.51 | 7.00 | 4.21 | $4.45 \mathrm{E}+0$ |
|  | ipMVSA | 11.34 | 8.26 | 3.34 | 1.94 | 1.01 | 30.23 | 30.38 | 16.29 | 10.30 | 6.39 | $8.14 \mathrm{E}-1$ |
|  | HyperCSI | 1.48 | 1.08 | 0.71 | 0.44 | 0.31 | 22.64 | 15.98 | 11.10 | 7.25 | 4.40 | $7.48 \mathrm{E}-2$ |

## Real data experiments

- Real hyperspectral imaging data experiments: Airborne Visible/Infrared Imaging Spectrometer (AVIRIS) taken over the Cuprite mining site, Nevada, in 1997 [AVIRIS'97].
- The number of sources for this dataset is estimated to be $N=9$ using an information-theoretic minimum description length (MDL) criterion [Lin'16-2].
- The proposed HyperCSI algorithm, along with the following two benchmark algorithms (for analyzing the hyperspectral imaging data), are used to process the AVIRIS data:
(1) MVC-NMF algorithm [Miao'07] (based on Craig's criterion);
(2) VCA algorithm [Nascimento'05] (based on pure-pixel assumption).

[^4]
## Real data experiments (con't)



Endmembers extracted by HyperCSI algorithm show better resemblance to their counterparts in library. For instance, the endmember of Alunite extracted by HyperCSI shows much clearer absorption feature than MVC-NMF and VCA, in the bands approximately from 2.3 to $2.5 \mu \mathrm{~m}$.

## Real data experiments (con't)

- The average RMS spectral angle error $\phi$ between endmember estimates and their corresponding library spectra, is used for quantitative comparison:

|  | HyperCSI | MVC-NMF | VCA |
| :---: | :---: | :---: | :---: |
| Muscovite | $\mathbf{3 . 0 3}$ | 3.96 | 4.54 |
| Alunite | 7.48 | $\mathbf{6 . 2 3}$ | 6.57 |
| Desert Varnish | 9.49 | $\mathbf{4 . 9 1}$ | 7.92 |
| Hematite | 7.83 | 12.94 | $\mathbf{7 . 2 4}$ |
| Montmorillonite | $\mathbf{4 . 8 4}$ | 7.44 | 6.59 |
| Kaolinite \#1 | 8.63 | $\mathbf{7 . 5 6}$ | $13.80(11.71)$ |
| Kaolinite \#2 | $\mathbf{7 . 3 9}$ | - | - |
| Buddingtonite | 6.55 | 8.16 | $\mathbf{6 . 4 6}$ |
| Chalcedony | $\mathbf{5 . 9 2}$ | 7.97 | 8.25 |
| Andradite | - | $\mathbf{7 . 4 3}$ | - |
| Average $\phi$ (degrees) | $\mathbf{6 . 8 0}$ | 7.40 | 8.12 |
| $T$ (seconds) | $\mathbf{0 . 1 2}$ | 988.67 | 5.40 |

- As the pure pixels may not be present in the selected subscene, the two Craig criterion based algorithms outperform VCA as expected.
- In terms of the computation time $T$, in spite of parallel processing not yet applied, HyperCSI is much faster than the other two algorithms.


## Real data experiments (con't)



Muscovite


Hematite


Kaolinite \#2


Alunite


Montmorillonite


Buddingtonite


Desert varnish


Kaolinite \#1


Chalcedony

The good performance of HyperCSI in the experiment also implies that the requirement of sufficient (i.e., $N(N-1)=72$ ) active pixels lying close to the hyperplanes of the actual endmembers' simplex, has been met for the considered hyperspectral scene.

## Summary

Based on the hyperplane representation for a simplest simplex, the presented HyperCSI algorithm has the following remarkable characteristics:
(1) Craig's simplex is reconstructed from $N(N-1)$ pixels (regardless of the data length $L$ ), without involving any simplex volume computations.
(2) It is reproducible (without involving random initialization and tuning of regularization parameters) and not data-dependent, regardless of the existence of pure pixels.
(3) Its superior performance over state-of-the-art methods has been demonstrated by analysis, simulations and real data experiments.
(4) It only involves simple linear algebraic computations, with a complexity $\mathcal{O}(N L)$ with or $\mathcal{O}\left(N^{2} L\right)$ without parallel implementation, thereby sustaining its practical applicability.

## Outline

(1) Part I: Fundamentals of Convex Optimization
2) Part II: Application in Hyperspectral Image Analysis: (Big Data Analysis and Machine Learning)
(3) Part III: Application in Wireless Communications (5G Systems)

- Subsection I: Outage Constrained Robust Transmit Optimization for Multiuser MISO Downlinks
- Subsection II: Outage Constrained Robust Hybrid Coordinated Beamforming for Massive MIMO Enabled Heterogeneous Cellular Networks


## Outline

(1) Part I: Fundamentals of Convex Optimization
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- Subsection I: Outage Constrained Robust Transmit Optimization for Multiuser MISO Downlinks
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## Outline (Subsection I)

(1) System Model and Problem Statement
(2) Convex Restriction Methods to the Rate Outage Constrained Problem: Bernstein-type Inequality
(3) Simulation Results
(9) Conclusions

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## 1. System Model and Problem Statement

- Multiuser multiple-input single-output (MISO) downlink: A practical scenario in wireless communications where one base station (BS) equipped with $N_{t}$ antennas sends independent messages to $K$ single-antenna users.



## 1. System Model and Problem Statement

- Transmit signal from BS:

$$
\begin{equation*}
\boldsymbol{x}(t)=\sum_{i=1}^{K} \boldsymbol{x}_{i}(t) \tag{43}
\end{equation*}
$$

- $\boldsymbol{x}_{i}(t) \in \mathbb{C}^{N_{t}}$ : information signal for user $i ; \boldsymbol{x}_{i}(t) \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{S}_{i}\right)$ with $\boldsymbol{S}_{i} \succeq \mathbf{0}$ denoting the signal covariance matrix.
- Assuming that $\operatorname{rank}\left(\boldsymbol{S}_{i}\right)=d, \boldsymbol{x}_{i}(t)$ can be expressed as [Vu07]

$$
\begin{equation*}
\boldsymbol{x}_{i}(t)=\sum_{k=1}^{d} \sqrt{\lambda_{k}\left(\boldsymbol{S}_{i}\right)} \boldsymbol{w}_{k} s_{i k}(t) \tag{44}
\end{equation*}
$$

- $\lambda_{k}\left(\boldsymbol{S}_{i}\right)$ : the $k$ th largest eigenvalue of $\boldsymbol{S}_{i}$;
- $s_{i k}(t) \sim \mathcal{C N}(0,1): k$ th independent data stream for user $i$;
- $\boldsymbol{w}_{k} \in \mathbb{C}^{N_{t}}$ : orthonormal eigenvectors of $\boldsymbol{S}_{i}$.
- When $d=1$, the transmit strategy for $\boldsymbol{x}_{i}(t)$ reduces to transmit beamforming.

[^5]
## 1. System Model and Problem Statement

- Received signal of user $i$ :

$$
\begin{equation*}
y_{i}(t)=\boldsymbol{h}_{i}^{H} \boldsymbol{x}(t)+n_{i}(t) . \tag{45}
\end{equation*}
$$

- $\boldsymbol{h}_{i} \in \mathbb{C}^{N_{t}}$ : the channel of user $i$;
- $n_{i}(t) \sim \mathcal{C N}\left(0, \sigma_{i}^{2}\right)$ : additive noise at the user $i$.
- Achievable rate of user $i$ (in bits $/ \mathrm{sec} / \mathrm{Hz}$ ), assuming single-user detection with perfect $\boldsymbol{h}_{i}$ at the receiver $i$ [Telatar99]:

$$
\begin{equation*}
\mathrm{R}_{i}\left(\left\{\boldsymbol{S}_{k}\right\}_{k=1}^{K} ; \boldsymbol{h}_{i}\right)=\log _{2}(1+\underbrace{\frac{\boldsymbol{h}_{i}^{H} \boldsymbol{S}_{i} \boldsymbol{h}_{i}}{\sum_{k \neq i}^{K} \boldsymbol{h}_{i}^{H} \boldsymbol{S}_{k} \boldsymbol{h}_{i}+\sigma_{i}^{2}}}_{\text {SINR }}), i=1, \ldots, K \tag{46}
\end{equation*}
$$

where SINR denotes the signal-to-interference-plus-noise ratio associated with user $i$.

## 1. System Model and Problem Statement

## Rate Constrained Problem under Perfect Channel State Information (CSI)

With the given CSI $\boldsymbol{h}_{1}, \ldots, \boldsymbol{h}_{K}$ that are known to the BS,

$$
\begin{align*}
\min _{\boldsymbol{S}_{1}, \ldots, \boldsymbol{S}_{K} \in \mathbb{H}^{N_{t}}} & \sum_{i=1}^{K} \operatorname{Tr}\left(\boldsymbol{S}_{i}\right)  \tag{47a}\\
\text { s.t. } & \mathrm{R}_{i}\left(\left\{\boldsymbol{S}_{k}\right\}_{k=1}^{K} ; \boldsymbol{h}_{i}\right) \geq r_{i}, \quad i=1, \ldots, K,  \tag{47b}\\
& \boldsymbol{S}_{1}, \ldots, \boldsymbol{S}_{K} \succeq \mathbf{0} \tag{47c}
\end{align*}
$$

where each $r_{i} \geq 0$ is the required information rate (target rate) for user $i$.

- Problem (47) can be reformulated as a convex semidefinite program (SDP), which is polynomial-time solvable [Bengtsson01] [Gershman10].

[^6]
## 1. System Model and Problem Statement

- Unfortunately, the BS cannot acquire perfect CSI $\boldsymbol{h}_{i}$ (used in the conventional formulation in (47)) due to imperfect channel estimation and limited feedback [Love08].
- CSI error model:

$$
\begin{equation*}
\boldsymbol{h}_{i}=\overline{\boldsymbol{h}}_{i}+\boldsymbol{e}_{i}, \quad i=1, \ldots, K \tag{48}
\end{equation*}
$$

where $\overline{\boldsymbol{h}}_{i} \in \mathbb{C}^{N_{t}}$ is the presumed channel at the BS , and $e_{i} \in \mathbb{C}^{N_{t}}$ is the channel error vector.

- Gaussian channel error model [Marco05] [Shenouda08] (suitable for imperfect channel estimation at BS):

$$
\begin{equation*}
\boldsymbol{e}_{i} \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{C}_{i}\right) \tag{49}
\end{equation*}
$$

for some known error covariance $\boldsymbol{C}_{i} \succeq \mathbf{0}$.

[^7]
## Effect of CSI errors on QoS requirement (information rate)

## Rate outage probabilities:

$$
\rho_{i} \triangleq \operatorname{Prob}\left\{\mathrm{R}_{i} \leq r_{i}\right\}, i=1, \ldots, K
$$

Consider a simulation example for $N_{t}=K=3$, target rates $r_{i}=r$, $\rho_{i} \leq \rho=0.1$ for all $i$, with the common target SINR value $\gamma \triangleq 2^{r}-1=11 \mathrm{~dB}$ (cf. (46)).

- 500 sets of $\left\{\overline{\boldsymbol{h}}_{i}\right\}_{i=1}^{K}$ are randomly generated with $\overline{\boldsymbol{h}}_{i} \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{I}_{N_{t}}\right)$;
- For each $\left\{\overline{\boldsymbol{h}}_{k}\right\}_{k=1}^{K}$,
- obtain $\left\{\hat{\boldsymbol{S}}_{k}\right\}_{k=1}^{K}$ by solving (47) (as if $\left\{\overline{\boldsymbol{h}}_{k}\right\}_{k=1}^{K}$ were perfect), and using 4 robust designs, probabilistic SOCP [Shenouda08] and Methods I-III (to be presented later) with $C_{i}=0.002 \boldsymbol{I}_{N_{t}}$;
- generate 10,000 Gaussian CSI errors $\boldsymbol{e}_{i} \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{C}_{i}\right), i=1, \ldots, K$, and then obtain the actual $\hat{\rho}_{i}=N_{i} / 10000$, where $N_{i}$ denotes the no. of error realizations for which

$$
\mathrm{R}_{i}\left(\left\{\hat{\boldsymbol{S}}_{k}\right\}_{k=1}^{K} ; \overline{\boldsymbol{h}}_{i}+\boldsymbol{e}_{i}\right) \leq r, i=1, \ldots, K
$$

[^8]
## Effect of CSI errors on QoS requirement (information rate)

- Histograms of actual satisfaction probabilities $\left\{1-\hat{\rho}_{i}\right\}_{i=1}^{K}$ over the 500 channel realizations:



## 1. System Model and Problem Statement

## Rate Outage Constrained Problem

Given rate requirements $r_{1}, \ldots, r_{K}>0$ and maximum tolerable outage probabilities $\rho_{1}, \ldots, \rho_{K} \in(0,1]$,

$$
\begin{align*}
\min _{\boldsymbol{S}_{1}, \ldots, \boldsymbol{S}_{K} \in \mathbb{H}^{\boldsymbol{N}_{t}}} & \sum_{i=1}^{K} \operatorname{Tr}\left(\boldsymbol{S}_{i}\right)  \tag{50a}\\
\text { s.t. } & \operatorname{Prob}\left\{\mathrm{R}_{i}\left(\left\{\boldsymbol{S}_{k}\right\}_{k=1}^{K} ; \overline{\boldsymbol{h}}_{i}+\boldsymbol{e}_{i}\right) \leq r_{i}\right\} \leq \rho_{i}, i=1, \ldots, K,  \tag{50b}\\
& \boldsymbol{S}_{1}, \ldots, \boldsymbol{S}_{K} \succeq \mathbf{0}, \tag{50c}
\end{align*}
$$

where $\mathrm{R}_{i}\left(\left\{\boldsymbol{S}_{k}\right\}_{k=1}^{K}, \overline{\boldsymbol{h}}_{i}+\boldsymbol{e}_{i}\right)$ is defined in (46).

- Problem (50) is hard to solve since rate outage probabilities in (50b) have no closed-form expressions and are unlikely to be efficiently computable in general.


## Outline (Subsection I)

(1) System Model and Problem Statement
(2) Convex Restriction Methods to the Rate Outage Constrained Problem: Bernstein-Type Inequality
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## 2. A Restriction Approach for Problem (50)

- The rate outage constraints in (50b) can be expressed as

$$
\begin{equation*}
\operatorname{Prob}\left\{\boldsymbol{e}_{i}^{H} \boldsymbol{Q}_{i} \boldsymbol{e}_{i}+2 \operatorname{Re}\left\{\boldsymbol{e}_{i}^{H} \boldsymbol{r}_{i}\right\}+s_{i}<0\right\} \leq \rho_{i}, \quad i=1, \ldots, K \tag{51}
\end{equation*}
$$

where for notational simplicity, $\boldsymbol{e}_{i} \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{I}_{N_{t}}\right)$ (originally denoting channel error), and

$$
\begin{align*}
\boldsymbol{Q}_{i} & =\boldsymbol{C}_{i}^{1 / 2}\left(\frac{1}{\gamma_{i}} \boldsymbol{S}_{i}-\sum_{k \neq i} \boldsymbol{S}_{k}\right) \boldsymbol{C}_{i}^{1 / 2}, \quad \boldsymbol{r}_{i}=\boldsymbol{C}_{i}^{1 / 2}\left(\frac{1}{\gamma_{i}} \boldsymbol{S}_{i}-\sum_{k \neq i} \boldsymbol{S}_{k}\right) \overline{\boldsymbol{h}}_{i},  \tag{52a}\\
s_{i} & =\overline{\boldsymbol{h}}_{i}^{H}\left(\frac{1}{\gamma_{i}} \boldsymbol{S}_{i}-\sum_{k \neq i} \boldsymbol{S}_{k}\right) \overline{\boldsymbol{h}}_{i}-\sigma_{i}^{2} \tag{52b}
\end{align*}
$$

in which

$$
\gamma_{i}=2^{r_{i}}-1 \quad(\text { target SINR for user } i(\text { cf. (46) }))
$$

corresponding to the rate requirement $r_{i}=\log _{2}\left(1+\gamma_{i}\right)$.

## 2. A Restriction Approach for Problem (50)

## Challenge 1

Let $\boldsymbol{e} \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{I}_{n}\right)$ and $(\boldsymbol{Q}, \boldsymbol{r}, s) \in \mathbb{H}^{n} \times \mathbb{C}^{n} \times \mathbb{R}$ be an arbitrary 3-tuple of (deterministic) variables. Find an efficiently computable convex function $f: \mathbb{H}^{n} \times \mathbb{C}^{n} \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
\operatorname{Prob}\left\{\boldsymbol{e}^{H} \boldsymbol{Q} \boldsymbol{e}+2 \operatorname{Re}\left\{\boldsymbol{e}^{H} \boldsymbol{r}\right\}+s<0\right\} \leq f(\boldsymbol{Q}, \boldsymbol{r}, s) \tag{53}
\end{equation*}
$$

- Once a function $f$ satisfying (53) is found,

$$
\begin{align*}
& f(\boldsymbol{Q}, \boldsymbol{r}, s) \leq \rho  \tag{54}\\
\Longrightarrow & \operatorname{Prob}\left\{\boldsymbol{e}^{H} \boldsymbol{Q} \boldsymbol{e}+2 \operatorname{Re}\left\{\boldsymbol{e}^{H} \boldsymbol{r}\right\}+s<0\right\} \leq \rho \tag{55}
\end{align*}
$$

- Constraint (54) gives a convex restriction of constraint (55), because the associated $\boldsymbol{Q}_{i}, \boldsymbol{r}_{i}, s_{i}$ defined by (52) are affine in $\boldsymbol{S}_{1}, \ldots, \boldsymbol{S}_{K}$, implying that both $f$ and (54) are convex with respect to decision variable $\boldsymbol{S}_{i}$.


## 2. Convex Restriction Methods (Bernstein-Type Inequality)

- Alternative expression for the outage probability constraint in (55):

$$
\operatorname{Prob}\left\{\boldsymbol{e}^{H} \boldsymbol{Q} \boldsymbol{e}+2 \operatorname{Re}\left\{\boldsymbol{e}^{H} \boldsymbol{r}\right\}+s \geq 0\right\} \geq 1-\rho
$$

## Lemma 2 (Bernstein-Type Inequality) [Bechar09]

Let $\boldsymbol{e} \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{I}_{n}\right)$, and let $\boldsymbol{Q} \in \mathbb{H}^{n}$ and $\boldsymbol{r} \in \mathbb{C}^{n}$ be given. Then, for any $\eta>0$,

$$
\begin{equation*}
\operatorname{Prob}\left\{\boldsymbol{e}^{H} \boldsymbol{Q} \boldsymbol{e}+2 \operatorname{Re}\left\{\boldsymbol{e}^{H} \boldsymbol{r}\right\} \geq \Upsilon(\eta)\right\} \geq 1-e^{-\eta} \tag{56}
\end{equation*}
$$

where $\Upsilon: \mathbb{R}_{++} \rightarrow \mathbb{R}$ is defined by

$$
\Upsilon(\eta)=\operatorname{Tr}(\boldsymbol{Q})-\sqrt{2 \eta} \sqrt{\|\boldsymbol{Q}\|_{F}^{2}+2\|\boldsymbol{r}\|_{2}^{2}}-\eta \lambda^{+}(\boldsymbol{Q})
$$

and $\lambda^{+}(\boldsymbol{Q})=\max \left\{\lambda_{\max }(-\boldsymbol{Q}), 0\right\}$.
[Bechar09] I. Bechar, "A Bernstein-type inequality for stochastic processes of quadratic forms of Gaussian variables," 2009, preprint, available on http://arxiv.org/abs/ 0909.3595.

## 2. Convex Restriction Methods (Bernstein-Type Inequality)

- A sufficient condition for achieving the outage constraint (55):

$$
f(\boldsymbol{Q}, \boldsymbol{r}, s)=e^{-\Upsilon^{-1}(-s)} \leq \rho
$$

or equivalently $\Upsilon(\ln (1 / \rho)) \geq-s$, i.e.,

$$
\begin{equation*}
\operatorname{Tr}(\boldsymbol{Q})-\sqrt{2 \ln (1 / \rho)} \sqrt{\|\boldsymbol{Q}\|_{F}^{2}+2\|\boldsymbol{r}\|_{2}^{2}}+\ln (\rho) \cdot \lambda^{+}(\boldsymbol{Q})+s \geq 0 \tag{57}
\end{equation*}
$$

## 2. Convex Restriction Methods (Bernstein-Type Inequality)

## Method II (Bernstein-Type Inequality)

A convex restriction approximation of problem (50):

$$
\begin{align*}
\min _{\substack{\boldsymbol{S}_{i} \in \mathbb{H}^{N_{t}, x_{i}, y_{i} \in \mathbb{R},} \\
i=1, \ldots, K}} & \sum_{i=1}^{K} \operatorname{Tr}\left(\boldsymbol{S}_{i}\right)  \tag{58a}\\
\text { s.t. } & \operatorname{Tr}\left(\boldsymbol{Q}_{i}\right)-\sqrt{2 \ln \left(1 / \rho_{i}\right)} \cdot x_{i}+\ln \left(\rho_{i}\right) \cdot y_{i}+s_{i} \geq 0, \forall i  \tag{58b}\\
& \left\|\left[\begin{array}{c}
\operatorname{vec}\left(\boldsymbol{Q}_{i}\right) \\
\sqrt{2} \boldsymbol{r}_{i}
\end{array}\right]\right\|_{2} \leq x_{i}, i=1, \ldots, K  \tag{58c}\\
& y_{i} \boldsymbol{I}_{N_{t}}+\boldsymbol{Q}_{i} \succeq \mathbf{0}, i=1, \ldots, K  \tag{58d}\\
& y_{1}, \ldots, y_{K} \geq 0, \quad \boldsymbol{S}_{1}, \ldots, \boldsymbol{S}_{K} \succeq \mathbf{0} \tag{58e}
\end{align*}
$$

where $\boldsymbol{Q}_{i}, \boldsymbol{r}_{i}$ and $s_{i}$ are defined in (52), $i=1, \ldots, K$.

## Outline (Subsection I)

(1) System Model and Problem Statement
(2) Convex Restriction Methods to the Rate Outage Constrained Problem: Bernstein-Type Inequality
(3) Simulation Results
(4) Conclusions

## 3. Simulation Results

## Simulation Setting (spatially i.i.d. Gaussian CSI errors):

- $N_{t}=K=3$;
- Users' noise powers: $\sigma_{1}^{2}=\cdots=\sigma_{K}^{2}=0.1$;
- Preset outage probabilities: $\rho_{1}=\cdots=\rho_{K}=0.1$;
- SINR requirements: $\gamma_{1}=\cdots=\gamma_{K} \triangleq \gamma\left(\right.$ recall that $\left.\gamma_{i}=2^{r_{i}}-1\right)$;
- Spatially i.i.d. Gaussian CSI errors $\boldsymbol{C}_{1}=\cdots=\boldsymbol{C}_{K}=0.002 \boldsymbol{I}_{N_{t}}$;
- In each simulation, 500 sets of the presumed channels $\left\{\overline{\boldsymbol{h}}_{i}\right\}_{i=1}^{K}$ are randomly and independently generated with $\overline{\boldsymbol{h}}_{i} \sim \mathcal{C N}\left(\mathbf{0}, \boldsymbol{I}_{N_{t}}\right)$;
- Performance comparisons with probabilistic SOCP [Shenouda08].
[Shenouda08] M. B. Shenouda and T. N. Davidson, "Probabilistically-constrained approaches to the design of the multiple antenna downlink," in Proc. 42nd Asilomar Conference 2008, Pacific Grove, October 26-29, 2008, pp. 1120-1124.


## 3. Simulation Results

Feasibility rate of the various methods, where Method II is Bernstein inequality based method.

Feasibility rate $\triangleq \frac{\text { no. of feasible channels }}{\text { no. of tested channels }} \times 100 \%$;
The lower the feasibility rate, the more conservative (implying higher transmit power in general) for the method under test.


## 3. Simulation Results

Transmit power performance of the various methods.


## 3. Simulation Results

- In the simulation, a "rank-1 solution" for $\left(\widehat{\boldsymbol{S}}_{1}, \ldots, \widehat{\boldsymbol{S}}_{K}\right)$ is obtained if $\frac{\lambda_{\max }\left(\widehat{\boldsymbol{S}}_{i}\right)}{\operatorname{Tr}\left(\hat{\boldsymbol{S}}_{i}\right)} \geq 0.9999$ for all $i$.
- Ratio of rank-one solution $\triangleq \frac{\text { no. of realizations yielding a rank-one solution }}{\text { no. of realizations yielding a feasible solution }}$.

Table 1: Ratios of rank-one solutions.

| $\rho$ | 0.1 |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $\gamma(\mathrm{~dB})$ | 3 | 7 | 11 | 15 |
| Method I | $464 / 464$ | $448 / 448$ | $404 / 404$ | $292 / 292$ |
| Method II | $489 / 489$ | $475 / 475$ | $441 / 441$ | $363 / 363$ |
| Method III | $488 / 488$ | $449 / 449$ | $372 / 372$ | $251 / 251$ |


| $\rho$ | 0.01 |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $\gamma(\mathrm{~dB})$ | 3 | 7 | 11 | 15 |
| Method I | $450 / 450$ | $424 / 424$ | $343 / 343$ | $225 / 225$ |
| Method II | $477 / 480$ | $463 / 463$ | $428 / 428$ | $322 / 322$ |
| Method III | $473 / 473$ | $418 / 418$ | $301 / 301$ | $124 / 124$ |

## Outline (Subsection I)

(1) System Model and Problem Statement
(2) Convex Restriction Methods to the Rate Outage Constrained Problem: Bernstein-Type Inequality
(3) Simulation Results
(4) Conclusions

## 4. Conclusions

- We considered the multiuser MISO downlink scenario with Gaussian CSI errors and studied a rate outage constrained optimization problem.
- Bernstein-type inequality based method for efficiently computable convex restriction of the probabilistic constraints using analytic tools from probability theory was presented [Wang'14].
- Simulation results demonstrated that Bernstein-type inequality based method significantly improve upon the existing state-of-the-art method [Shenouda08] in terms of both computational complexity and solution accuracy.

[^9]
## Outline

(1) Part I: Fundamentals of Convex Optimization
(2) Part II: Application in Hyperspectral Image Analysis: (Big Data Analysis and Machine Learning)
(3) Part III: Application in Wireless Communications (5G Systems)

- Subsection I: Outage Constrained Robust Transmit Optimization for Multiuser MISO Downlinks
- Subsection II: Outage Constrained Robust Hybrid Coordinated Beamforming for Massive MIMO Enabled Heterogeneous Cellular Networks


## Outline (Subsection II)

(1) System Model and Problem Formulation
(2) Proposed Outage Constrained HyCoBF Design
(1) Analog Beamforming Algorithm
(2) Digital Robust CoBF Design
(3) Simulation Results
(4) Conclusions

## Outline (Subsection II)

(1) System Model and Problem Formulation
(2) Proposed Outage Constrained HyCoBF Design
(1) Analog Beamforming Algorithm
(2) Digital Robust CoBF Design
(3) Simulation Results
(4) Conclusions

## 1. System Model and Problem Formulation

- Massive MIMO enabled two-tier heterogeneous network (HetNet): A macrocell base station (MBS) equipped with large-scale $N_{\text {MBS }}$ antennas, and a femtocell base station (FBS) equipped with $N_{\text {FBS }}$ antennas, serve $K$ single-antenna macrocell user equipments (MUEs) and $J$ single-antenna femtocell user equipments (FUEs), respectively.



## 1. System Model and Problem Formulation

- Hybrid coordinated beamforming (HyCoBF) structure at MBS for the HetNet, with $N_{\mathrm{RF}}$ radio frequency (RF) chains satisfying $N_{\mathrm{MBS}} \gg N_{\mathrm{RF}}$ $\geq K$, and $N_{\mathrm{RF}} \times N_{\mathrm{MBS}}$ analog phase shifters [Molisch'16].

[Molisch'16] A. F. Molisch, V. V. Ratnam, S. Han, Z. Li, S. Nguyen, S. Li, and K. Haneda, "Hybrid beamforming for massive MIMO-A survey," http://arxiv.org/pdf/1609.05078v1.pdf, Sep. 2016.


## 1. System Model and Problem Formulation

- Let $\mathbf{F} \in \mathbb{C}^{N_{\text {MBS }} \times N_{\text {MBS }}}$ be the $N_{\text {MBS-point }}$ discrete Fourier transform (DFT) matrix (codebook), and

$$
\begin{equation*}
\mathcal{B} \triangleq\left\{\mathbf{b} \in\left\{1, \ldots, N_{\mathrm{MBS}}\right\}^{N_{\mathrm{RF}}} \mid b_{i} \neq b_{j}, \forall i \neq j\right\} \tag{59}
\end{equation*}
$$

collecting all the $\binom{N_{\mathrm{MBS}}}{N_{\mathrm{RF}}}$ different combinations.

- The selected analog beamforming matrix $\mathbf{V}$, consisting of $N_{\mathrm{RF}}$ distinct columns of $\mathbf{F}$, and the RF switch matrix $\boldsymbol{\Lambda}(\mathbf{b})$ can be expressed as

$$
\begin{align*}
& \boldsymbol{\Lambda}(\mathbf{b})=\left[\boldsymbol{e}\left(b_{1}\right), \ldots, \boldsymbol{e}\left(b_{N_{\mathrm{RF}}}\right)\right] \in \mathbb{R}^{N_{\mathrm{MBS}} \times N_{\mathrm{RF}}}, \mathbf{b} \in \mathcal{B}, \\
& \mathbf{V}=\mathbf{F} \boldsymbol{\Lambda}(\mathbf{b}) \in \mathbb{C}^{N_{\mathrm{MBS}} \times N_{\mathrm{RF}}} \tag{60}
\end{align*}
$$

where $\boldsymbol{e}\left(b_{i}\right)$ denotes the $b_{i}$-th unit column vector.

## 1. System Model and Problem Formulation

- Received signal of MUE $k$ :

$$
\begin{align*}
y_{M k} & =\mathbf{h}_{M M k}^{H} \mathbf{V} \mathbf{w}_{k} s_{M k}+\sum_{l=1, l \neq k}^{K} \mathbf{h}_{M M k}^{H} \mathbf{V} \mathbf{w}_{l} s_{M l} \\
& +\sum_{j=1}^{J} \mathbf{h}_{F M k}^{H} \mathbf{u}_{j} s_{F j}+n_{M k} \tag{61}
\end{align*}
$$

- $s_{M k}, s_{F j}$ : transmit signals with unit power intended for MUE $k$ and FUE $j$, respectively; $n_{M k} \sim \mathcal{C N}\left(0, \sigma_{M k}^{2}\right)$ (AWGN);
- $\mathbf{h}_{M M k} \in \mathbb{C}^{N_{\text {MBS }}}, \mathbf{h}_{F M k} \in \mathbb{C}^{N_{\mathrm{FBS}}}$ : channels from MBS and FBS to MUE $k$, respectively;
- $\mathbf{w}_{k} \in \mathbb{C}^{N_{\mathrm{RF}}}, \mathbf{u}_{j} \in \mathbb{C}^{N_{\mathrm{FBS}}}$ : baseband beamforming vectors for MUE $k$ and FUE $j$, respectively.
- Signal-to-interference-plus-noise ratio (SINR) of MUE $k$ :

$$
\begin{equation*}
\operatorname{SINR}_{M k}=\frac{\left|\mathbf{h}_{M M k}^{H} \mathbf{V} \mathbf{w}_{k}\right|^{2}}{\sum_{l=1, l \neq k}^{K}\left|\mathbf{h}_{M M k}^{H} \mathbf{V} \mathbf{w}_{l}\right|^{2}+\sum_{j=1}^{J}\left|\mathbf{h}_{F M k}^{H} \mathbf{u}_{j}\right|^{2}+\sigma_{M k}^{2}} \tag{62}
\end{equation*}
$$

## 1. System Model and Problem Formulation

- Received signal of FUE $j$ :

$$
\begin{align*}
y_{F j}= & \mathbf{h}_{F F j}^{H} \mathbf{u}_{j} s_{F j}+\sum_{m=1, m \neq j}^{J} \mathbf{h}_{F F j}^{H} \mathbf{u}_{m} s_{F m} \\
& +\sum_{k=1}^{K} \mathbf{h}_{M F j}^{H} \mathbf{V} \mathbf{w}_{k} s_{M k}+n_{F j} \tag{63}
\end{align*}
$$

- $\mathbf{h}_{M F j} \in \mathbb{C}^{N_{M B S}}, \mathbf{h}_{F F j} \in \mathbb{C}^{N_{\mathrm{FBS}}}$ : channels from MBS and FBS to FUE $j$, respectively; $n_{F j} \sim \mathcal{C N}\left(0, \sigma_{F j}^{2}\right)$.
- SINR of FUE $j$ :

$$
\begin{equation*}
\operatorname{SINR}_{F j}=\frac{\left|\mathbf{h}_{F F j}^{H} \mathbf{u}_{j}\right|^{2}}{\sum_{m=1, m \neq j}^{J}\left|\mathbf{h}_{F F j}^{H} \mathbf{u}_{m}\right|^{2}+\sum_{k=1}^{K}\left|\mathbf{h}_{M F j}^{H} \mathbf{V} \mathbf{w}_{k}\right|^{2}+\sigma_{F j}^{2}} \tag{64}
\end{equation*}
$$

## 1. System Model and Problem Formulation

- Imperfect CSI model [Wang'14]:

$$
\begin{align*}
& \mathbf{h}_{M M k}=\hat{\mathbf{h}}_{M M k}+\mathbf{e}_{M M k}, \mathbf{h}_{M F j}=\hat{\mathbf{h}}_{M F j}+\mathbf{e}_{M F j},  \tag{65a}\\
& \mathbf{h}_{F F j}=\hat{\mathbf{h}}_{F F j}+\mathbf{e}_{F F j}, \mathbf{h}_{F M k}=\hat{\mathbf{h}}_{F M k}+\mathbf{e}_{F M k} \tag{65b}
\end{align*}
$$

- $\hat{\mathbf{h}}_{M M k}, \hat{\mathbf{h}}_{M F j} \in \mathbb{C}^{N_{\mathrm{MBS}}}$, and $\hat{\mathbf{h}}_{F F j}, \hat{\mathbf{h}}_{F M k} \in \mathbb{C}^{N_{\mathrm{FBS}}}$ : channel estimates that are known to MBS and FBS;
- $\mathbf{e}_{M M k}, \mathbf{e}_{M F j} \in \mathbb{C}^{N_{M B S}}, \mathbf{e}_{F F j}, \mathbf{e}_{F M k} \in \mathbb{C}^{N_{\mathrm{FBS}}}$ : Gaussian CSI errors

$$
\begin{aligned}
& \quad \mathbf{e}_{M M k} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{C}_{M M k}\right), \mathbf{e}_{M F j} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{C}_{M F j}\right), \\
& \quad \mathbf{e}_{F F j} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{C}_{F F j}\right), \mathbf{e}_{F M k} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{C}_{F M k}\right), \\
& \text { where } \mathbf{C}_{M M k} \succeq \mathbf{0}, \mathbf{C}_{M F j} \succeq \mathbf{0} \text { and } \mathbf{C}_{F F j} \succeq \mathbf{0}, \mathbf{C}_{F M k} \succeq \mathbf{0}
\end{aligned}
$$

[Wang'14] K.-Y. Wang, A. M.-C. So, T.-H. Chang, W.-K. Ma, and Chong-Yung Chi, "Outage constrained robust transmit optimization for multiuser MISO downlinks: Tractable approximations by conic optimization," IEEE Trans. Signal Processing, vol. 62, no. 21, pp. 5690-5705, November 2014.

## 1. System Model and Problem Formulation

## Outage Constrained Robust HyCoBF Problem

Given $\gamma_{M k}$ and $\gamma_{F j}$ for the target SINRs of MUEs and FUEs, respectively, and the associated outage probabilities $\rho_{M k}$ and $\rho_{F j}$,

$$
\begin{align*}
\min _{\mathbf{v},\left\{\mathbf{w}_{k}\right\},\left\{\mathbf{u}_{j}\right\}} & \sum_{k \in \mathcal{I}_{K}}\left\|\mathbf{V}_{\mathbf{w}_{k}}\right\|^{2}+\sum_{j \in \mathcal{I}_{J}}\left\|\mathbf{u}_{j}\right\|^{2}  \tag{67a}\\
\text { s.t. } & \operatorname{Pr}\left(\operatorname{SINR}_{M k} \geq \gamma_{M k}\right) \geq 1-\rho_{M k}, \forall k \in \mathcal{I}_{K}=\{1, \ldots, K\}  \tag{67b}\\
& \operatorname{Pr}\left(\operatorname{SINR}_{F j} \geq \gamma_{F j}\right) \geq 1-\rho_{F j}, \forall j \in \mathcal{I}_{J}=\{1, \ldots, J\}  \tag{67c}\\
& \mathbf{V} \in\{\mathbf{F} \boldsymbol{\Lambda}(\mathbf{b}) \mid \mathbf{b} \in \mathcal{B}\}(\operatorname{cf.}(2)) . \tag{67d}
\end{align*}
$$

- Solving the robust HyCoBF design problem (67) is hard, due to nonconvex probabilistic constraints (67b), (67c) [Nemirovski'06] and analog beam selection constraint (67d).
[Nemirovski'06] A. Nemirovski and A. Shapiro, "Convex approximations of chance constrained programs," SIAM J. Optim., vol. 17, no. 4, pp. 969-996, 2006.


## Outline (Subsection II)

(1) System Model and Problem Formulation
(2) Proposed Outage Constrained HyCoBF Design
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### 2.1. Analog Beamforming Algorithm

- A new beam selection criterion named power ratio maximization (PRM), by maximizing the ratio of the total channel power of MUEs to the total intercell interference channel power from MBS to FUEs:

$$
\begin{align*}
& \mathbf{V}^{\star}=\mathbf{F} \boldsymbol{\Lambda}\left(\mathbf{b}^{\star}\right) \quad(\mathrm{cf.} .(60))  \tag{68}\\
& \mathbf{b}^{\star}=\arg \max _{\mathbf{b} \in \mathcal{B}}\left\{\mathcal{J}(\mathbf{b}) \triangleq \frac{\left\|\widehat{\mathbf{H}}_{M M}^{H} \mathbf{F} \boldsymbol{\Lambda}(\mathbf{b})\right\|_{F}^{2}}{\left\|\widehat{\mathbf{H}}_{M F}^{H} \mathbf{F} \boldsymbol{\Lambda}(\mathbf{b})\right\|_{F}^{2}}\right\}, \text { (PRM criterion) } \tag{69}
\end{align*}
$$

where $\mathcal{B}$ was defined in (59), $\widehat{\mathbf{H}}_{M M} \triangleq\left[\hat{\mathbf{h}}_{M M 1}, \ldots, \hat{\mathbf{h}}_{M M K}\right]$ and $\widehat{\mathbf{H}}_{M F} \triangleq\left[\hat{\mathbf{h}}_{M F 1}, \ldots, \hat{\mathbf{h}}_{M F J}\right]$.

- A low-complexity beam selection algorithm for solving (69) is proposed, motivated by the idea of single most likely replacement (SMLR) detector used in seismic deconvolution for detecting a Bernoulli-Gaussian signal with nonzero magnitudes [Mendel'83].

[^10]
### 2.1. Analog Beamforming Algorithm

- Low complexity and guaranteed convergence: Computing $\mathcal{J}(\mathbf{b})$ by (69) $N_{\mathrm{RF}} \times\left(N_{\mathrm{MBS}}-N_{\mathrm{RF}}\right)$ times per iteration.


## Beam Selection Algorithm-PRM

(1) Given $\widehat{\mathbf{H}}_{M M}, \widehat{\mathbf{H}}_{M F}, \mathcal{I}_{N_{\mathrm{MBS}}}=\left\{1, \ldots, N_{\mathrm{MBS}}\right\}$;
(2) Initialize $\mathbf{b}=\left(b_{1}, \ldots, b_{N_{\mathrm{RF}}}\right) \in \mathcal{B}$, and $\mathcal{S}=\mathcal{I}_{N_{\mathrm{MBS}}} \backslash\left\{b_{1}, \ldots, b_{N_{\mathrm{RF}}}\right\}$;
(3) Compute $\mathcal{J}(\mathbf{b})$ by (69);
(4) repeat
(5) Obtain $j_{\ell}=\arg \max _{j \in \mathcal{S}} \mathcal{J}\left(\mathbf{b}_{\ell}(j)\right), \ell=1, \ldots, N_{\mathrm{RF}}$, where $\mathbf{b}_{\ell}(j) \triangleq\left(b_{1}, \ldots, b_{\ell-1}, j, b_{\ell+1}, \ldots, b_{N_{\mathrm{RF}}}\right)$;
(6) Obtain $l=\arg \max \left\{\mathcal{J}\left(\mathbf{b}_{\ell}\left(j_{\ell}\right)\right), \ell=1, \ldots, N_{\mathrm{RF}}\right\}$;
(7) If $\mathcal{J}\left(\mathbf{b}_{l}\left(j_{l}\right)\right)>\mathcal{J}(\mathbf{b})$, update $\mathbf{b}:=\mathbf{b}_{l}\left(j_{l}\right), \mathcal{S}:=\mathcal{I}_{N_{\mathrm{MBS}}} \backslash\left\{b_{1}, \ldots, b_{N_{\mathrm{RF}}}\right\}$, $\mathcal{J}(\mathbf{b}):=\mathcal{J}\left(\mathbf{b}_{l}\left(j_{l}\right)\right)$;
(8) until $\mathcal{J}\left(\mathbf{b}_{l}\left(j_{l}\right)\right) \leq \mathcal{J}(\mathbf{b})$.
(9) Output selected analog beamforming matrix $\mathbf{V}^{\star}=\mathbf{F} \boldsymbol{\Lambda}(\mathrm{b})$.

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(1) System Model and Problem Formulation
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### 2.2. Conservative Approximate CoBF Solution

- Let $\mathbf{W}_{k}=\mathbf{w}_{k} \mathbf{w}_{k}^{H}$ and $\mathbf{U}_{j}=\mathbf{u}_{j} \mathbf{u}_{j}^{H}$, and define

$$
\begin{align*}
& \mathbf{B}_{k} \triangleq \gamma_{M k}^{-1} \mathbf{V}^{\star} \mathbf{W}_{k}\left(\mathbf{V}^{\star}\right)^{H}-\sum_{l \neq k}^{K} \mathbf{V}^{\star} \mathbf{W}_{l}\left(\mathbf{V}^{\star}\right)^{H} ; \mathbf{D} \triangleq-\sum_{j=1}^{J} \mathbf{U}_{j} ;  \tag{70a}\\
& \mathbf{F}_{j} \triangleq \gamma_{F j}^{-1} \mathbf{U}_{j}-\sum_{l \neq j}^{J} \mathbf{U}_{l} ; \quad \mathbf{G} \triangleq-\sum_{k=1}^{K} \mathbf{V}^{\star} \mathbf{W}_{k}\left(\mathbf{V}^{\star}\right)^{H} ;  \tag{70b}\\
& \mathbf{Q}_{M M k} \triangleq \mathbf{C}_{M M k}^{1 / 2} \mathbf{B}_{k} \mathbf{C}_{M M k}^{1 / 2} ; \quad \mathbf{Q}_{F M k} \triangleq \mathbf{C}_{F M k}^{1 / 2} \mathbf{D} \mathbf{C}_{F M k}^{1 / 2} ;  \tag{70c}\\
& \mathbf{Q}_{F F j} \triangleq \mathbf{C}_{F F j}^{1 / 2} \mathbf{F}_{j} \mathbf{C}_{F F j}^{1 / 2} ; \quad \mathbf{Q}_{M F j} \triangleq \mathbf{C}_{M F j}^{1 / 2} \mathbf{G C}_{M F j}^{1 / 2} ;  \tag{70d}\\
& \mathbf{r}_{M M k} \triangleq \mathbf{C}_{M M k}^{1 / 2} \mathbf{B}_{k} \hat{\mathbf{h}}_{M M k} ; \quad \mathbf{r}_{F M k} \triangleq \mathbf{C}_{F M k}^{1 / 2} \mathbf{D} \hat{\mathbf{h}}_{F M k} ;  \tag{70e}\\
& \mathbf{r}_{F F j} \triangleq \mathbf{C}_{F F j}^{1 / 2} \mathbf{F}_{j} \hat{\mathbf{h}}_{F F j} ; \mathbf{r}_{M F j} \triangleq \mathbf{C}_{M F j}^{1 / 2} \mathbf{G}_{M F j} ;  \tag{70f}\\
& c_{M k} \triangleq \hat{\mathbf{h}}_{M M k}^{H} \mathbf{B}_{k} \hat{\mathbf{h}}_{M M k}+\hat{\mathbf{h}}_{F M k}^{H} \mathbf{D} \hat{\mathbf{h}}_{F M k}-\sigma_{M k}^{2} ;  \tag{70~g}\\
& c_{F j} \triangleq \hat{\mathbf{h}}_{F F j}^{H} \mathbf{F}_{j} \hat{\mathbf{h}}_{F F j}+\hat{\mathbf{h}}_{M F j}^{H} \mathbf{G} \hat{\mathbf{h}}_{M F j}-\sigma_{F j}^{2} . \tag{70h}
\end{align*}
$$

### 2.2. Conservative Approximate CoBF Solution

## Outage Constrained Digital CoBF: Conservative Approximation

Applying semidefinite relaxation (SDR) (i.e., replacing $\mathbf{w}_{k} \mathbf{w}_{k}^{H}$ by $\mathbf{W}_{k} \succeq \mathbf{0}$ and $\mathbf{u}_{j} \mathbf{u}_{j}^{H}$ by $\mathbf{U}_{j} \succeq \mathbf{0}$ ) to problem (67) yields

$$
\begin{align*}
& \min _{\left\{\mathbf{W}_{k}\right\},\left\{\mathbf{U}_{j}\right\}} \sum_{k \in \mathcal{I}_{K}} \operatorname{Tr}\left(\mathbf{V}^{\star} \mathbf{W}_{k}\left(\mathbf{V}^{\star}\right)^{H}\right)+\sum_{j \in \mathcal{I}_{J}} \operatorname{Tr}\left(\mathbf{U}_{j}\right)  \tag{71a}\\
& \text { s.t. } \operatorname{Pr}\left\{\boldsymbol{\delta}_{1}^{H} \mathbf{Q}_{M M k} \boldsymbol{\delta}_{1}+\boldsymbol{\delta}_{2}^{H} \mathbf{Q}_{F M k} \boldsymbol{\delta}_{2}+2 \operatorname{Re}\left\{\boldsymbol{\delta}_{1}^{H} \mathbf{r}_{M M k}\right\}+\right. \\
&\left.2 \operatorname{Re}\left\{\boldsymbol{\delta}_{2}^{H} \mathbf{r}_{F M k}\right\}+c_{M k} \geq 0\right\} \geq 1-\rho_{M k}, \quad \forall k \in \mathcal{I}_{K},  \tag{71b}\\
& \operatorname{Pr}\{ \boldsymbol{\delta}_{1}^{H} \mathbf{Q}_{M F j} \boldsymbol{\delta}_{1}+\boldsymbol{\delta}_{2}^{H} \mathbf{Q}_{F F j} \boldsymbol{\delta}_{2}+2 \operatorname{Re}\left\{\boldsymbol{\delta}_{1}^{H} \mathbf{r}_{M F j}\right\}+ \\
&\left.2 \operatorname{Re}\left\{\boldsymbol{\delta}_{2}^{H} \mathbf{r}_{F F j}\right\}+c_{F j} \geq 0\right\} \geq 1-\rho_{F j}, \quad \forall j \in \mathcal{I}_{J},  \tag{71c}\\
& \mathbf{W}_{k} \succeq \mathbf{0}, \forall k \in \mathcal{I}_{K}, \mathbf{U}_{j} \succeq \mathbf{0}, \forall j \in \mathcal{I}_{J} . \tag{71d}
\end{align*}
$$

where $\boldsymbol{\delta}_{1} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{I}_{N_{\mathrm{MBS}}}\right), \boldsymbol{\delta}_{2} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{I}_{N_{\mathrm{FBS}}}\right)$.

### 2.2. Conservative Approximate CoBF Solution

- Define

$$
\begin{gather*}
g_{1}\left(\boldsymbol{\delta}_{1}, \mathbf{Q}_{M M k}, \mathbf{r}_{M M k}\right) \triangleq \boldsymbol{\delta}_{1}^{H} \mathbf{Q}_{M M k} \boldsymbol{\delta}_{1}+2 \operatorname{Re}\left\{\mathbf{r}_{M M k}^{H} \boldsymbol{\delta}_{1}\right\},  \tag{72}\\
g_{2}\left(\boldsymbol{\delta}_{2}, \mathbf{Q}_{F M k}, \mathbf{r}_{F M k}\right) \triangleq \boldsymbol{\delta}_{2}^{H} \mathbf{Q}_{F M k} \boldsymbol{\delta}_{2}+2 \operatorname{Re}\left\{\mathbf{r}_{F M k}^{H} \boldsymbol{\delta}_{2}\right\} \tag{73}
\end{gather*}
$$

- Alternative expression for the outage probability constraint in (71b):

$$
\begin{align*}
\operatorname{Pr}\left\{g_{1}\left(\boldsymbol{\delta}_{1}, \mathbf{Q}_{M M k}, \mathbf{r}_{M M k}\right)+g_{2}\left(\boldsymbol{\delta}_{2}, \mathbf{Q}_{F M k}, \mathbf{r}_{F M k}\right)\right. & \left.+c_{M k} \geq 0\right\} \\
& \geq 1-\rho_{M k} \tag{74}
\end{align*}
$$

- (71c) can also be equivalently re-expressed as the same form as (74).
- An extension form of the Bernstein-type inequality [Bechar'09] is presented in Lemma 1 below for finding a conservative convex approximation to (74).

[^11]
### 2.2. Conservative Approximate CoBF Solution

## Lemma 1: Extension Form of the Bernstein-type Inequality

Given $\boldsymbol{\delta}_{1} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{I}_{N_{\mathrm{MBS}}}\right), \boldsymbol{\delta}_{2} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{I}_{N_{\mathrm{FBS}}}\right), \mathbf{Q}_{M M k} \in \mathbb{H}^{N_{\mathrm{MBS}}}$, $\mathbf{Q}_{F M k} \in \mathbb{H}^{N_{\mathrm{FBS}}}, \mathbf{r}_{F M k} \in \mathbb{C}^{N_{\mathrm{FBS}}}$ Then, the following inequality holds:

$$
\begin{align*}
& \operatorname{Pr}\left\{g_{1}\left(\boldsymbol{\delta}_{1}, \mathbf{Q}_{M M k}, \mathbf{r}_{M M k}\right)+g_{2}\left(\boldsymbol{\delta}_{2}, \mathbf{Q}_{F M k}, \mathbf{r}_{F M k}\right) \geq\right. \\
& \left.\Upsilon\left(\ln \left(1 / \rho_{M k}\right) \mid \mathbf{Q}_{M M k}, \mathbf{r}_{M M k}, \mathbf{Q}_{F M k}, \mathbf{r}_{F M k}\right)\right\} \geq 1-\rho_{M k}, \forall k \in \mathcal{I}_{K}, \tag{75}
\end{align*}
$$

where $\Upsilon: \mathbb{R}_{++} \rightarrow \mathbb{R}$ is defined as

$$
\begin{align*}
& \Upsilon\left(\ln \left(1 / \rho_{M k}\right) \mid \mathbf{Q}_{M M k}, \mathbf{r}_{M M k}, \mathbf{Q}_{F M k}, \mathbf{r}_{F M k}\right) \triangleq \\
& \operatorname{Tr}\left(\mathbf{Q}_{M M k}\right)+\operatorname{Tr}\left(\mathbf{Q}_{F M k}\right)-\ln \left(1 / \rho_{M k}\right) \cdot \lambda^{+}\left(\mathbf{Q}_{M M k}, \mathbf{Q}_{F M k}\right) \\
& -\alpha_{M k} \sqrt{\left\|\mathbf{Q}_{M M k}\right\|_{F}^{2}+2\left\|\mathbf{r}_{M M k}\right\|^{2}+\left\|\mathbf{Q}_{F M k}\right\|_{F}^{2}+2\left\|\mathbf{r}_{F M k}\right\|^{2}} \tag{76}
\end{align*}
$$

in which $\lambda^{+}\left(\mathbf{Q}_{M M k}, \mathbf{Q}_{F M k}\right) \triangleq \max \left\{\lambda_{\max }\left(-\mathbf{Q}_{M M k}\right), \lambda_{\max }\left(-\mathbf{Q}_{F M k}\right), 0\right\}$ and $\alpha_{M k}=\sqrt{2 \ln \left(1 / \rho_{M k}\right)}$.

By Lemma 1, $\Upsilon\left(\ln \left(1 / \rho_{M k}\right) \mid \mathbf{Q}_{M M k}, \mathbf{r}_{M M k}, \mathbf{Q}_{F M k}, \mathbf{r}_{F M k}\right)+c_{M k} \geq 0$ is an approximate deterministic conservative convex constraint to (71b).

### 2.2. Conservative Approximate CoBF Solution

## Robust Digital CoBF Problem (Centralized Solution)

By Lemma 1, problem (71) can be approximated as a convex semidefinite programming (SDP) [Luo'10]:

$$
\begin{align*}
\min _{\substack{\left.k \\
\mathbf{W}_{k}\right\},\left\{\mathbf{U}_{j}\right\}  \tag{77a}\\
\mathbf{t}_{M}, \mathbf{t}_{F}}} & \left.\sum_{k \in \mathcal{I}_{K}} \operatorname{Tr}\left(\mathbf{V}^{\star} \mathbf{W}_{k}\left(\mathbf{V}^{\star}\right)^{H}\right)\right)+\sum_{j \in \mathcal{I}_{J}} \operatorname{Tr}\left(\mathbf{U}_{j}\right)  \tag{77b}\\
\text { s.t. } & \left(\left\{\mathbf{W}_{k}\right\},\left\{\mathbf{U}_{j}\right\}, \mathbf{t}_{M}\right) \in \mathcal{C}_{M},  \tag{77c}\\
& \left(\left\{\mathbf{W}_{k}\right\},\left\{\mathbf{U}_{j}\right\}, \mathbf{t}_{F}\right) \in \mathcal{C}_{F},  \tag{77d}\\
& \mathbf{W}_{k} \succeq \mathbf{0}, \forall k \in \mathcal{I}_{K}, \mathbf{U}_{j} \succeq \mathbf{0}, \forall j \in \mathcal{I}_{J},
\end{align*}
$$

- $\mathcal{C}_{M}$ and $\mathcal{C}_{F}$ are approximate conservative convex constraint sets to (71b) and (71c) (cf. (79) and (80) in Appendix A), respectively;
- $\mathbf{t}_{M} \in \mathbb{R}^{3 K}, \mathbf{t}_{F} \in \mathbb{R}^{3 J}$ are auxiliary variables.
[Luo'10] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems," IEEE Signal Process. Mag., vol. 27, pp. 20-34, May 2010.


## Outline (Subsection II)

(1) System Model and Problem Formulation
(2) Proposed Outage Constrained HyCoBF Design
(1) Analog Beamforming Algorithm
(2) Digital Robust CoBF Design
(3) Simulation Results
(4) Conclusions

## 3. Simulation Results

## Simulation Setting:

- Users' AWGN powers: $\sigma_{k}^{2}=\sigma_{j}^{2}=\sigma^{2}, \forall k \in \mathcal{I}_{K}, \forall j \in \mathcal{I}_{J}$;
- Target SINRs: $\gamma_{M k}=\gamma_{F j}=\gamma, \forall k, j$;
- SINR outage probabilities: $\rho_{M k}=\rho_{F j}=\rho$;
- CSI error covariance matrices: $\mathbf{C}_{M M k}=\mathbf{C}_{M F j}=\varepsilon^{2} \mathbf{I}_{N_{M B S}}$, $\mathbf{C}_{F F j}=\mathbf{C}_{F M k}=\varepsilon^{2} \mathbf{I}_{N_{\mathrm{FBS}}}$;
- The performance evaluations were performed using CVX for the proposed HyCoBF design.
- The digital CoBF of the proposed robust HyCoBF design reduces to full digital (FD) CoBF as $N_{\mathrm{MBS}}=N_{\mathrm{RF}}$.


## 3. Simulation Results (Power Performance)

- $N_{\mathrm{MBS}}=16, K=4, N_{\mathrm{FBS}}=J=2, N_{\mathrm{RF}}=\{4,8,16\} ; \rho=0.1$, $\varepsilon^{2}=\{0.001,0.002\}$.



## 3. Simulation Results (Power Performance)

- $N_{\mathrm{MBS}}=64, K=4, N_{\mathrm{FBS}} \in\{2,4\}, N_{\mathrm{RF}}=\{4,8,16\}, J=2 ; \rho=0.1$, $\varepsilon^{2}=0.002$.



## 3. Simulation Results (Satisfaction Probability)

- $N_{\mathrm{MBS}}=16, N_{\mathrm{RF}}=K=4 ; N_{\mathrm{FBS}}=J=2 ; \rho=0.1, \gamma=9 \mathrm{~dB}$ and $\varepsilon^{2}=0.002$.






## 3. Simulation Results (Rank-one Solution)

- In simulation, a "rank-one solution" for $\mathbf{W}_{k}^{\star}$ and $\mathbf{U}_{j}^{\star}$ is obtained if the following conditions hold:

$$
\begin{equation*}
\frac{\lambda_{\max }\left(\mathbf{W}_{k}^{\star}\right)}{\operatorname{Tr}\left(\mathbf{W}_{k}^{\star}\right)} \geq 0.9999, \frac{\lambda_{\max }\left(\mathbf{U}_{j}^{\star}\right)}{\operatorname{Tr}\left(\mathbf{U}_{j}^{\star}\right)} \geq 0.9999, k \in \mathcal{I}_{\mathcal{K}}, j \in \mathcal{I}_{\mathcal{J}} . \tag{78}
\end{equation*}
$$

- Counts of rank-one solutions and all the feasible solutions for each simulation case for $\varepsilon^{2} \in\{0.01,0.002\}$ and $\gamma \in\{1,5,9,13\} \mathrm{dB}$.

| $\varepsilon^{2}$ | 0.01 |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $\gamma(\mathrm{~dB})$ | 1 | 5 | 9 | 13 |
| Robust FD | $(447,447)$ | $(396,396)$ | $(286,286)$ | $(102,102)$ |
| HyCoBF-PRM | $(418,418)$ | $(250,250)$ | $(63,63)$ | $(35,35)$ |


| $\varepsilon^{2}$ | 0.002 |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $\gamma(\mathrm{~dB})$ | 1 | 5 | 9 | 13 |
| Robust FD | $(490,490)$ | $(481,481)$ | $(460,460)$ | $(401,401)$ |
| HyCoBF-PRM | $(413,413)$ | $(402,402)$ | $(336,336)$ | $(191,191)$ |

## Outline (Subsection II)

(1) System Model and Problem Formulation
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## 4. Conclusions

- We have introduced an outage probability constrained robust HyCoBF design for a massive MIMO enabled HetNet in the presence of Gaussian CSI errors, by solving a nonconvex total transmit power minimization problem.
- The robust HyCoBF design consists of:
- The analog beamforming at MBS is a newly devised low-complexity beam selection scheme by maximizing the ratio of MUEs' channel power to FUEs' interference channel power, given a DFT matrix codebook.
- The digital beamforming at MBS and FBS is a conservative approximate $C o B F$ solution (by SDR technique and an extended Bernstein-type inequality).


## 4. Conclusions

- Simulation results have demonstrated that the proposed robust HyCoBF design algorithm can yield promising performance and, most importantly, it can achieve comparable performance to the FD beamforming scheme with much fewer RF chains.
- Recently, a distributed implementation for the CoBF solution using ADMM in the proposed HyCoBF has been finished [ $\mathrm{Xu}^{\prime} 17$ ].
[Xu'17] G.-X. Xu, C.-H. Lin, W.-G. Ma, S.-Z. Chen, and Chong-Yung Chi, "Outage constrained robust hybrid coordinated beamforming for massive MIMO enabled heterogeneous cellular networks," IEEE Access, vol. 5, pp. 13601-13616, Mar. 2017.


## Appdendix A

The convex constraint by using Lemma 1 for (71b) can be expressed in the following form:

$$
\begin{align*}
\mathcal{C}_{M} \triangleq\{ & \left(\left\{\mathbf{W}_{k}\right\},\left\{\mathbf{U}_{j}\right\}, \mathbf{t}_{M}\right) \mid \\
& \operatorname{Tr}\left(\mathbf{Q}_{M M k}\right)+\operatorname{Tr}\left(\mathbf{Q}_{F M k}\right)+\ln \left(\rho_{M k}\right) y_{M k}+c_{M k}- \\
& \alpha_{M k}\left\|\left[x_{M M k}, x_{F M k}\right]^{T}\right\| \geq 0, \forall k \in \mathcal{I}_{K} \\
& \left\|\left[\begin{array}{c}
\operatorname{vec}\left(\mathbf{Q}_{M M k}\right) \\
\sqrt{2} \mathbf{r}_{M M k}
\end{array}\right]\right\| \leq x_{M M k}, \forall k \in \mathcal{I}_{K}  \tag{79}\\
& \left\|\left[\begin{array}{c}
\operatorname{vec}\left(\mathbf{Q}_{F M k}\right) \\
\sqrt{2} \mathbf{r}_{F M k}
\end{array}\right]\right\| \leq x_{F M k}, \forall k \in \mathcal{I}_{K} \\
& y_{M k} \mathbf{I}_{N_{\mathrm{MBS}}}+\mathbf{Q}_{M M k} \succeq \mathbf{0}, \\
& \left.y_{M k} \mathbf{I}_{N_{\mathrm{FBS}}}+\mathbf{Q}_{F M k} \succeq \mathbf{0}, y_{M k} \geq 0, \forall k \in \mathcal{I}_{K}\right\}
\end{align*}
$$

where $\mathbf{t}_{M} \triangleq\left[x_{M M 1}, \ldots, x_{M M K}, x_{F M 1}, \ldots, x_{F M K}, y_{M 1}, \ldots, y_{M K}\right]^{T} \in \mathbb{R}^{3 K}$ are the introduced auxiliary variables.

## Appdendix A

Similarly, (71c) can be expressed in the following form:

$$
\begin{align*}
\mathcal{C}_{F} \triangleq\{ & \left(\left\{\mathbf{W}_{k}\right\},\left\{\mathbf{U}_{j}\right\}, \mathbf{t}_{F}\right) \mid \\
& \operatorname{Tr}\left(\mathbf{Q}_{F F j}\right)+\operatorname{Tr}\left(\mathbf{Q}_{M F j}\right)+\ln \left(\rho_{F j}\right) y_{F j}+c_{F j}- \\
& \beta_{F j}\left\|\left[x_{F F j}, x_{M F j}\right]^{T}\right\| \geq 0, \forall j \in \mathcal{I}_{J} \\
& \left\|\left[\begin{array}{c}
\operatorname{vec}\left(\mathbf{Q}_{F F j}\right) \\
\sqrt{2} \mathbf{r}_{F F j}
\end{array}\right]\right\| \leq x_{F F j}, \forall j \in \mathcal{I}_{J}  \tag{80}\\
& \left\|\left[\begin{array}{c}
\operatorname{vec}\left(\mathbf{Q}_{M F j}\right) \\
\sqrt{2} \mathbf{r}_{M F j}
\end{array}\right]\right\| \leq x_{M F j}, \forall j \in \mathcal{I}_{J} \\
& y_{F j} \mathbf{I}_{N_{M B S}}+\mathbf{Q}_{M F j} \succeq \mathbf{0}, \\
& \left.y_{F j} \mathbf{I}_{N_{\mathrm{FBS}}}+\mathbf{Q}_{F F j} \succeq \mathbf{0}, y_{F j} \geq 0, \forall j \in \mathcal{I}_{J}\right\}
\end{align*}
$$

where $\mathbf{t}_{F} \triangleq\left[x_{F F 1}, \ldots, x_{F F J}, x_{M F 1}, \ldots, x_{M F J}, y_{F 1}, \ldots, y_{F J}\right]^{T} \in \mathbb{R}^{3 J}$ collects all the auxiliary variables in the derivation of (80).


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